SOME HYPERBOLIC $OUT(F_N)$-GRAPHS AND NONUNIQUE ERGODICITY OF VERY SMALL F_N-TREES

by

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ABSTRACT

We define a new graph on which $Out(F_N)$ acts and show that it is hyperbolic. Also we give a new proof, based on an argument by Bestvina and Fujiwara, that the Free Factor Graph satisfies Weak Proper Discontinuity (WPD), and show that the Intersection Graph satisfies WPD as well.

Furthermore, in joint work with Patrick Reynolds, we construct nonuniquely ergodic, nongeometric, arational F_N-trees.

CONTENTS

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[1]The rest of this chapter appears in *Geometriae Dedicata*, published online 06 December 2013 as *Hyperbolicity of the Cyclic Spitting Graph*. It is reproduced here with kind permission from Springer Science and Business Media.

[2]Most of this chapter appears as a preprint at http://arxiv.org/abs/1311.1771. This is joint work with Patrick Reynolds.

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CHAPTER 1

HYPERBOLIC $OUT(F_N)$-GRAPHS

1.1 An Introduction

Let N be a positive integer, and $S = \{a_1, \ldots, a_N\}$ be a set with N letters. We define the *free group* of rank N, denoted F_N as follows. As a set, F_N consists of all strings in the symbols $a_1^{\pm 1}, \ldots, a_N^{\pm 1}$ modulo the relation that $wa_ia_i^{-1}u \sim wu$ and $wa_i^{-1}a_iu \sim wu$ for all $1 \leq i \leq N$. The group operation in F_N is concatenation of strings, and the identity element is the empty string. Write $aa \cdots a$ (the concatenation of n copies of a) as a^n.

Example 1.1. *Let $N = 2$ and $S = \{a, b\}$. Then in F_2, we have $aba \cdot b^{-1}ab = abab^{-1}ab$, and $ab^2 \cdot b^{-1}a = aba$.*

It is clear that with these operations, F_N forms a group. Indeed, the inverse of $a_{i_1}^{\epsilon} a_{i_2}^{\epsilon} \cdots a_{i_k}^{\epsilon}$ is easily checked to be $a_{i_k}^{-\epsilon} \cdots a_{i_1}^{-\epsilon}$. Let us note some useful facts about F_N.

Theorem 1.1 (Nielsen-Schreier)**.** *Every subgroup of F_N is itself a free group.*

Theorem 1.2 (Howson's Theorem)**.** *The intersection of two finitely generated subgroups of F_N is itself finitely generated.*

As F_N is a group, we can talk about the group of all automorphisms of F_N, denoted $Aut(F_N)$. There is a special normal subgroup $Inn(F_N) \triangleleft Aut(F_N)$, called the *inner automorphisms*, which consists of all those automorphisms given by conjugation (i.e., automorphisms of the form $x \mapsto gxg^{-1}$ for some fixed $g \in F_N$). Note that $Inn(F_N) \cong F_N$.

As $Inn(F_N)$ is normal, it makes sense to consider the quotient $Out(F_N) = Aut(F_N)/Inn(F_N)$. This group, the *Outer Automorphism Group* of F_N will be our main object of study. The goal of this part is to study $Out(F_N)$ via its action on various, well-behaved metric spaces. To this end, we will define several metric spaces on which $Out(F_N)$ acts by isometries.

1.2 Culler-Vogtmann Outer Space

In an attempt to study $Out(F_N)$ geometrically, Culler and Vogtmann invented the space CV_N, *Outer Space*, in [16]. CV_N is equipped with an asymmetric "metric" and $Out(F_N)$ acts by isometries on CV_N. CV_N is contractible and point-stabilizers of this action are finite. The definition of CV_N mirrors that of the Teichmuller space of a surface S on which the mapping class group $Mod(S)$ acts by isometries. The focus of this thesis is not the mapping class group, so we refer the reader to [17] for more information on surfaces, $Mod(S)$, and Teichmuller space.

1.2.1 F_N as the Fundamental Group of a Graph

For our purposes, a *graph* will be a 1-dimensional simplicial complex. Graphs can either be finite or infinite. Unless stated otherwise, all of our graphs will be connected.

Example 1.2. *A* rose with N-petals *is a specific example of a graph which has precisely one vertex and N edges. We denote such a graph by R_N. One can also consider roses with an infinite number of edges.*

Proposition 1.1. *Let R be a rose. $\pi_1(R)$ is free of rank equal to the number of edges in R (see Figure 1.1).*

Proof. Since R is homeomorphism to a wedge sum of copies of S^1, by Van-Kampen's theorem, the fundamental group is the one we assert it to be. □

Proposition 1.2. *Let G be any graph. Then $\pi_1(G)$ is free.*

Proof. Choose a maximal tree $T \subset G$. Collapsing T to a point gives a homotopy equivalence from G to a rose R, since all vertices in G are contained in T. □

We say that G has *rank* N if $\pi_1(G) \cong F_N$.

1.2.2 The Definition of Culler-Vogtmann Outer Space

Here we construct CV_N. For more details, see [4]. Fix once and for all a rose R with an identification of $\pi_1(R) = F_N$ (i.e., each edge of R is labelled by a basis element $a_1, \ldots, a_N$ for F_N). All of our graphs in this section are finite, 3-valent (that is, every vertex has at least 3 incident edges), and rank N.

A *marking* of a graph G is a homotopy equivalence $g : R \to G$. We usually think about a marking as a labeling of the edges in $G \setminus \{\text{some maximal tree}\}$ by a basis for F_N. A *metric* on a graph G is a function $l : E(G) \to [0, \infty]$, where $E(G)$ is the set of edges of G, and where the set of edges e where $l(e) = 0$ must be a forest in G. We say l is *volume 1* if $\sum_{e \in E(G)} l(e) = 1$.

Define an equivalence relation on the set of triples (G, g, l), where G is a graph, g is a marking, and l is a metric of volume 1, as follows: $(G, g, l) \sim (G', g', l')$ if there exists an isometry $\phi : G \to G'$ such that $\phi \circ g \simeq g'$ (where we identify edge with an interval in $\mathbb{R}$ with length $l(e)$, and where we think of edges with length 0 as being collapsed to a point. This is why we insist that the set of edges of length zero must be a forest). See Figure 1.2 for an example.

As a set CV_N is the set of triples (G, g, l), we need a topology on CV_N:

Each topological type of marked graph gives a simplex with some missing faces by varying the the lengths of the edges (because of the condition that $\sum_{e \in E(G)} l(e) = 1$). We define a poset structure on nonmetric marked graphs by declaring that $(G, g) \geq (G', g')$ if there exists a homotopy equivalence $f : G \to G'$ collapsing some forest in G such that $f \circ g \simeq g'$.

Another less precise way to say this is that the marked graph G' is obtained from G by collapsing a forest.

The topology on CV_N is that induced by the geometric realization of the poset defined above, where we think about each marked-nonmetric graph as corresponding to a simplex with missing faces of dimension $e - 1$, where e is the number of edges in the graph. Each point in the simplex corresponds to a metric of volume 1 on the graph. It is left to the reader to show that this is, in fact, a simplicial complex (with missing faces). Details can be found in [4].

Each $\phi \in Out(F_N)$ is represented by a homotopy equivalence $\varphi : R \to R$. Define a right action of $Out(F_N)$ on CV_N by $(G, g, l)\phi = (G, g \circ \varphi, l)$. When we define a metric on CV_N, we will see that $Out(F_N)$ acts by isometries.

1.2.3 The Metric on CV_N

This definition of the metric on CV_N is motivated by the standard metric on Teichmuller space. Unfortunately (or fortunately perhaps) the notion of distance we define is not symmetric; that is, $d(G, G')$ is not usually equal to $d(G', G)$. So it is not a metric in the usual sense, but for lack of a better term, we shall simply refer to it as an asymmetric metric. (The author suggests calling it an *asymmetric*, but no one else seems keen on this idea).

Given two marked graphs (G, g) and (G', g'), a *difference of marking* is a homotopy equivalence $f : G \to G'$ which is linear on edges and such that $f \circ g \simeq g'$. Given a difference of marking f, we define $\sigma(f)$ to be the optimal Lipschitz constant of f (i.e., the maximal slope over all the edges of G).

Define

$$d(G, G') = \log \inf \sigma(f)$$

where the infimum is taken over all differences of marking $f : G \to G'$. d satisfies all of the normal properties of a metric, aside from symmetry. The proofs of these properties are elementary. For details, see [4].

In fact, the infimum is always realized by Arzela-Ascoli. Suppose $h : G \to G'$ is a map realizing above infimum. A pair of directions (d_1, d_2) at a vertex is called a *turn*. A turn is *illegal* if $h(d_1) = h(d_2)$ as directions in G', and *legal* otherwise. A closed loop α is *legal* if all turns crossed by α are legal. Abusing notation, we may also use α to mean a conjugacy class in F_N, in which case, $l_G(\alpha)$ is the length of the reduced loop in G representing α.

Proposition 1.3 (see [4]). *$d(G, G') = \log \max_\alpha \frac{l_{G'}(\alpha)}{l_G(\alpha)}$, where the maximum is taken over all conjugacy classes α.*

Note that, if $h : G \to G'$ is as above, we see that the maximum is achieved only on legal loops. Since each legal loop in G contains an embedded legal circle, figure-eight, or dumbell, it suffices to check the value of the above quotient on subgraphs of these three types.

Thus, the above proposition enables us to compute distance in CV_N by simply checking the quotient $\frac{l_{G'}(\alpha)}{l_G(\alpha)}$ on all loops of one of the three forms listed above, and seeing where it is largest. Since in a finite graph there are finitely many subgraphs of these types, to compute distance in CV_N, one need only make finitely many computations.

The proof of the following proposition is an easy consequence of the definitions of marked graphs and the metric on CV_N.

Proposition 1.4. *$Out(F_N)$ acts on CV_N by isometries.*

1.3 Gromov Hyperbolic Metric Spaces

From now on, all our metric spaces will be assumed to be geodesic. A metric space (X, d) is *δ-hyperbolic* if there exists some $\delta > 0$ so that for any three points $p, q, r \in X$, any geodesic segment pq lies in a δ-neighborhood of $qr \cup pr$.

If X is δ-hyperbolic for some δ, we will often just say that X is *hyperbolic*.

Example 1.3. *The hyperbolic plane (the upper half plane in $\mathbb{R}^2$ with the metric $\frac{dx^2+dy^2}{y^2}$) is $\ln(1+\sqrt{2})$-hyperbolic.*

For an example more related to the rest of this thesis, let us define a special type of metric space which will turn out to be 0-hyperbolic. An $\mathbb{R}$-*tree* is a connected metric space (X, d) such that for any $x, y \in X$, there exists a unique path $f : [0, d(x, y)] \to X$ with $f(0) = x$ and $f(d(x, y)) = y$, and this path is geodesic (an isometric embedding). Simplicial trees are examples of $\mathbb{R}$-trees. However, not every $\mathbb{R}$-tree is a simplicial complex.

Example 1.4. *Every $\mathbb{R}$-tree is 0-hyperbolic. Indeed, every geodesic triangle is a tripod.*

CV_N are not hyperbolic. Indeed, there are Euclidean planes quasi-isometrically embedded in CV_N. For example, consider the following automorphisms of F_4: $\phi : a \mapsto bab, b \mapsto ab, c \mapsto c, d \mapsto d$ and $\psi : a \mapsto a, b \mapsto b, c \mapsto dcd, d \mapsto cd$.

ϕ and ψ obviously commute. Fix a graph $R \in CV_N$, which is the rose on 4 petals with standard marking. Our claim is that orbit of R under $\langle \phi, \psi \rangle$ is a quasi-isometrically embedded copy of $\mathbb{Z}^2$ in CV_N.

Note that $d(R, \phi^n\psi^m R) = d(R, \psi^m R) + d(\psi^n R, \phi^n\psi^m R)$. Since ψ is represented by a train track map on R (see [3]), it follows that $d(R, \psi^m R) = md(R, \psi R)$. Furthermore, $d(\psi^m R, \phi^n\psi^m R) = d(\psi^m R, \psi^m\phi^n R) = nd(R, \phi^n R)$. The claim follows.

1.4 Hyperbolicity of the Cyclic Splitting Graph[2]

Let S be a hyperbolic surface, perhaps with nonempty boundary or punctures. The *curve graph* $\mathcal{C}(S)$ associated to S is a simplicial graph defined by taking vertices to be homotopy classes of essential simple close curves in S, and where two vertices are joined by an edge if they can be represented by disjoint curves in the surface.

A celebrated theorem of Masur and Minsky (see [33]) is:

Theorem (Masur-Minsky)**.** *The curve graph $\mathcal{C}(S)$ is δ-hyperbolic.*

The mapping class group of S, denoted by $Mod(S)$, acts on $\mathcal{C}(S)$ in the obvious manner.

Fix an $n \geq 3$ for the rest of the paper. In an attempt to mirror the study of surfaces and their mapping class groups, there have been several complexes defined on which $Out(F_n)$, the group of outer automorphisms of the rank n free group, acts. One such complex is the *free factor complex*, denoted FF_n, whose vertices are conjugacy classes of proper free factors and which has the structure of a simplicial complex given by the poset structure on the conjugacy classes of free factors of F_n. Bestvina and Feighn [8] proved:

[2]The rest of this chapter appears in *Geometriae Dedicata*, published online 06 December 2013 as *Hyperbolicity of the Cyclic Spitting Graph.* It is reproduced here with kind permission from Springer Science and Business Media.

Theorem (Bestvina-Feighn)**.** *The free factor complex FF_n is δ-hyperbolic.*

Furthermore, Bestvina-Reynolds [10] and Hamenstaedt [23] have recently given a description of the Gromov boundary of FF_n.

Another analogue for the curve graph is the *free splitting graph*, FS_n, whose vertices are one-edge free splittings of F_n, and where two vertices are joined by an edge if the two splittings admit a common refinement. Handel and Mosher [24] showed:

Theorem (Handel-Mosher)**.** *The free splitting complex FS_n is δ-hyperbolic.*

More recently, another proof using sphere systems in $\#_{i=1}^{n} S^1 \times S^2$ was given by Hilion and Horbez [26]. A recent paper of Kapovich and Rafi [29] shows that the hyperbolicity of the free splitting complex implies the hyperbolicity of the free factor complex. The outline of their argument goes as follows: the authors define an auxiliary complex FB_n called the *complex of free bases*, whose vertices are conjugacy classes of free bases of F_n, and where two conjugacy classes of bases are adjacent if they have representatives which share an element. They show that FB_n and FF_n are quasi-isometric, so it suffices to show the hyperbolicity of FB_n, which they do by applying a consequence of Bowditch's work (see Theorem 2.1 below) in [11] to the natural map $FS_n' \to FB_n$, where FS_n' is the barycentric subdivision of FS_n.

1.4.1 The Cyclic Splitting Graph

In this paper, we define another analogue of the curve graph for surfaces, the *cyclic splitting graph*, and show that it is hyperbolic using the technology from [29].

The cyclic splitting graph is the simplicial graph whose vertices are free splittings of F_n, and where two free splittings X and Y are connected by an edge if either (1) they are commonly refined (as in FS_n) or (2) if there is an element w in the vertex groups of X and Y and a $\mathbb{Z}$-splitting T with edge group $\langle w \rangle$ such that T can be obtained from X and Y by "folding" $\langle w \rangle$ over the trivial edge groups. There is a natural map $FS_n \to FZ_n$.

It is worth noting why one might wish to examine such a complex. By work of Stallings (see [34]) a simple closed curve c in a surface S gives a $\mathbb{Z}$-splitting of $\pi_1(S) = A *_{\langle c \rangle} B$ or $\pi_1(S) = A *_{\langle c \rangle}$, and conversely. The definition of FZ_n is constructed so as to mimic this way of thinking of $\mathcal{C}(S)$.

Our main theorem is:

Theorem A. *The cyclic splitting graph FZ_n is δ-hyperbolic.*

In further work, the author wishes to describe the elements of $Out(F_n)$ which act hyperbolically on FZ_n. In particular, a description of these elements would show that the natural map $FZ_n \to FF_n$ is not an $Out(F_n)$-equivariant quasi-isometry.

1.5 A Bowditch Hyperbolicity Condition for Graphs

A *graph* is a connected 1-dimensional simplicial complex. If X and Y are graphs, a *graph map* is a continuous map $f : X \to Y$ such that vertices map to vertices. As always, the vertex set of a graph X is denoted by $V(X)$, and the edge set by $E(X)$. From now on, whenever considering a (connected) simplicial complex Z as a metric space, we mean the 1-skeleton of Z with the simplicial metric. We denote a geodesic path from a vertex x to a vertex y by $[x, y]$.

In [11], Bowditch develops a criterion for a graph to be δ-hyperbolic. Similar criteria were applied in the Masur-Minsky proof that the curve graph is hyperbolic. Bowditch defines for constants $B_1, B_2 > 0$ a (B_1, B_2)-*thin triangles structure* in a graph X, which is a set of paths g_{xy} between any $x, y \in X$ satisfying some "thinness" conditions (see [29] for details). Bowditch proves the following useful condition for checking hyperbolicity in [11]:

Theorem 1.3 (Bowditch). *Suppose X is a connected graph. If there are B_1, B_2 such that X has a (B_1, B_2)-thin triangles structure, then there are $\delta > 0$ and $H > 0$ (depending on B_1, B_2) such that X is δ-hyperbolic, and every geodesic path from x to y in X is H-close to g_{xy}.*

The following theorem, which is proved as a corollary of the above theorem in [29], will be the main technical tool:

Theorem 1.4 (Kapovich-Rafi). *Suppose X and Y are connected graphs, X is δ-hyperbolic, and $f : X \to Y$ is L-Lipschitz for some $L \geq 0$. Suppose there is $S \subseteq V(X)$ such that*

1. *$f(S) = V(Y)$*
2. *S is D-dense in $V(X)$ for some $D \geq 0$.*
3. *There is an $M > 0$ such that if $x, y \in S$ with $d(f(x), f(y)) \leq 1$ then $diam(f[x, y]) \leq M$.*

Then Y is δ'-hyperbolic for some δ'.

1.6 The Free Splitting Complex

A *tree* is a simply connected graph. An *action* of a group G on a tree T, denoted $G \curvearrowright T$, is a homomorphism from G to the group of simplicial automorphisms of T. An action $G \curvearrowright T$ is called *minimal* if there is no proper G invariant subtree of T.

Let F_n denote the free group on n-generators. Recall from Bass-Serre theory that a minimal action $F_n \curvearrowright T$ with trivial edge stabilizers gives a a graph of groups decomposition of F_n with trivial edge groups (and hence a free splitting), and conversely. We shall often refer to the action $F_n \curvearrowright T$ as a free splitting, as there will be no confusion. A *k-edge splitting* refers to a free splitting whose associated graph of groups decomposition consists of k edges. Two splittings $F_n \curvearrowright T$ and $F_n \curvearrowright T'$ are *equivalent* if there exists an F_n equivariant homeomorphism $T \to T'$.

An equivariant map $f : T \to T'$ between minimal F_n-trees is called a *collapse map* if the preimage of any point is connected.

Define the *free splitting complex of* F_n, denoted FS_n as follows. For a more complete discussion, see [24]. A vertex of FS_n is an equivalence class of one-edge free splittings. Two vertices $X, Y \in V(FS_n)$ are connected by an edge if there exists a two edge splitting T and F_n-equivariant collapse maps $T \to X$ and $T \to Y$. We say T is a *common refinement* of X and Y. A k-simplex in FS_n is a collection of $k+1$ vertices $X_1, \ldots, X_{k+1}$ such that there exists a $k+1$ edge splitting T and F_n-equivariant collapse maps $T \to X_i$ for each $i = 1, \ldots, k+1$.

Denote by FS'_n the 1-skeleton of the barycentric subdivision of FS_n. This is a simplicial graph whose vertices correspond to free splittings of F_n, and where there is an edge between vertices T and T' if there is an equivariant collapse map $T \to T'$ or $T' \to T$. The complexes FS_n and FS'_n are finite dimensional, connected, and have an action of $Out(F_n)$ by simplicial automorphisms such that the quotient is compact (see [24]).

1.7 Folding Paths in FS_n

For a general definition of folding paths in FS'_n, see [24]. We need only special types of folding paths between splittings in Culler-Vogtmann Outer Space [16] which are also discussed in [29], but we will give an explanation here as well following the treatment there.

Let T be a tree. Vertices of valence ≥ 3 are *natural vertices*, and connected components of $T \setminus \{\text{natural vertices}\}$ are *natural edges*.

A *rose* R_n is a graph with one vertex and n-edges. Given an identification $F_n = \pi_1(R_n)$, a *marking* of a graph Γ is a homotopy equivalence $f : \Gamma \to R_n$. Two markings $f : \Gamma \to R_n$

and $f' : \Gamma' \to R_n$ are equivalent if there is a homeomorphism $\phi : \Gamma' \to \Gamma$ such that $f'\phi \simeq f'$. In particular, an equivalence class of markings gives an isomorphism $\pi_1(\Gamma) \to F_n$.

Let β be a basis for F_n. As in [29], define a *β-graph* to be a graph Γ with a function $\mu : E(\Gamma) \to \beta \cup \beta^{-1}$ such that if e is an oriented edge of Γ and $\bar{e}$ denotes the edge with the opposite orientation, then $\mu(\bar{e}) = \mu(e)^{-1}$. Let R_β be the rose whose (oriented) edges are labelled by elements of β and their inverses. This labeling gives an identification of F_n with $\pi_1(R_\beta)$.

The labeling of a β-graph Γ determines a map $\Gamma \to R_\beta$ by sending each edge of Γ to the edge of R_β with the same label. In particular, if this map $\Gamma \to R_\beta$ is a homotopy equivalence, then the labeling of Γ gives a marking.

Remark 1.1. *A marking of Γ corresponds to an action of F_n on the universal cover $\tilde{\Gamma}$, and hence to a point in FS'_n. Equivalent markings define the same vertex in FS'_n. We will use Γ to denote the vertex in FS'_n determined by the marking, hopefully without any confusion.*

1.7.1 Stallings Folds

Let Γ be a β-graph such that there exists two edges e_1 and e_2 with the same initial vertex and such that $\mu(e_1) = \mu(e_2)$. We obtain another β-graph Γ' by identifying the edges e_1 and e_2, and labeling the resulting edge by $\mu(e_1) = \mu(e_2)$. This is called a *Stallings fold* (see [40]). There is a quotient map $\Gamma \to \Gamma'$ which is call a *fold map*. Note that if e_1 and e_2 have distinct terminal vertices, then $\Gamma \to \Gamma'$ is a homotopy equivalence.

Suppose that Γ is a β-graph and that the labeling gives a marking $\Gamma \to R_\beta$. If we have two edges in Γ with the same initial vertex and label, we can construct another graph Γ' from Γ by a Stallings fold, and the marking $\Gamma \to R_\beta$ factors as $\Gamma \to \Gamma' \to R_\beta$. Furthermore, the map $\Gamma' \to R_\beta$ is again a marking.

1.7.2 Maximal Folds

Suppose Γ is a β-graph and that there exist two edges e_1 and e_2 in Γ with the same initial edge and such that $\mu(e_1) = \mu(e_2)$. Let $\hat{e}_1$ and $\hat{e_2}$ be natural edges containing e_1 and e_2. Then $\hat{e}_1$ and $\hat{e}_2$ contain maximal initial segments $\tilde{e}_1$ and $\tilde{e}_2$ which are labeled by the same word in β. Therefore, we can obtain another graph Γ' by identifying the segments $\tilde{e}_1$ and $\tilde{e}_2$. We say Γ' is obtained by a *maximal Stallings fold* or just a maximal fold.

1.7.3 Foldable Maps and Handel-Mosher Folding Paths

Let Γ be a β-graph, and $f : \Gamma \to R_\beta$ given by the labeling is a marking. We say that the map f is *foldable* if

1. For every vertex v of valence 2, the edges e_1 and e_2 with initial vertex v have $\mu(e_1) \neq \mu(e_2)$.

2. For every natural vertex v, there exist three edges e_1, e_2, and e_3 with the same initial vertex v such that $\mu(e_1)$, $\mu(e_2)$, and $\mu(e_3)$ are pairwise unequal.

There are a few important properties about foldable maps and maximal folds which we need:

- A map $\Gamma \to R_\beta$ is foldable in the sense above if and only if the corresponding map between F_n-trees $\tilde{\Gamma} \to \tilde{R_\beta}$ is foldable in the sense of Handel-Mosher in [24].

- If Γ is a β-graph and $\Gamma \to R_\beta$ is foldable, and if Γ' is obtained from Γ by a maximal fold, the induced map $\Gamma' \to R_\beta$ is foldable.

- If $\Gamma \to \Gamma'$ is a maximal fold, then $d_{FS'_n}(\Gamma, \Gamma') \leq 2$.

- If $\Gamma \to R_\beta$ is a marking then there exists a finite sequence of maximal folds

$$\Gamma = \Gamma_0 \to \Gamma_1 \to \ldots \to \Gamma_N = R_\beta$$

The proofs of these are elementary and found in [24]. We will also need the following result:

Theorem 1.5 (Handel-Mosher [24])**.** *The path in FS'_n given by connecting each Γ_i and Γ_{i+1} by an edge path of length ≤ 2 is an unparametrized quasi-geodesic in FS'_n.*

1.8 The Cyclic Splitting Graph FZ_n

First, let $F_n \curvearrowright T$ be a free splitting. Let v be a vertex of T, and let G_v be the stabilizer of v in $F_n = G$. Suppose $w \in G_v$ is nontrivial and let $\langle w \rangle$ denote the cyclic subgroup generated by w. Construct a new F_n-tree T as follows: choose an edge e with initial vertex v. Then for every $\gamma \in G$, identify γe with its orbit under the conjugate $\langle \gamma w \gamma^{-1} \rangle \subseteq G_{\gamma v}$.

The resulting tree T' corresponds to a graph of groups decomposition with an edge group $\langle w \rangle$. In particular, if T is a one edge free splitting, then T' is a one edge splitting with edge group $\mathbb{Z}$. Such a splitting is called a $\mathbb{Z}$-splitting of F_n. See Figures 1.3 and 1.4.

We say T' is obtained from T by an equivariant *edge fold*. The natural map $T \to T'$ is called an *edge folding map*.

Example 1.5. *Suppose $F_4 = \langle a, b, c, d \rangle$. Consider the one-edge free splitting $A * B$ given by $A = \langle a, b \rangle$ and $B = \langle c, d \rangle$. Then the one-edge $\mathbb{Z}$-splitting $A *_{\langle [a,b] \rangle} \langle B, [a, b] \rangle$ is obtained from $A * B$ by an edge fold.*

It is a theorem of Bestvina and Feighn (see Lemma 4.1 in [5]) that any $\mathbb{Z}$-splitting can be "unfolded."

Theorem 1.6 (Bestvina-Feighn [5])**.** *Let Γ be a graph of groups decomposition of the free group F_n with cyclic edge groups. Then either all of the edge groups of Γ are trivial or there exists an edge e with stabilizer $G_e \cong \mathbb{Z}$ and a vertex v which is an endpoint of e such that the inclusion $i : G_e \to G_v$ has as its image a free factor of G_v and furthermore, for any edge e' incident at v, the image of $G_{e'} \to G_v$ lies in a complementary free factor.*

In particular, Theorem 5.1 generalizes an earlier theorem of Shenitzer, Stallings, and Swarup (see [37], [41], [43]) that any $\mathbb{Z}$-splitting $A *_{\mathbb{Z}} B$ is obtained by edge folding from a free splitting as in the above example.

Define a graph FZ_n, the *cyclic splitting graph of F_n*, as follows. The vertices of FZ_n are one-edge free splittings of F_n up to equivariant homeomorphism. Free splittings X and Y are connected by an edge if

- there exists a two-edge splitting and F_n-equivariant collapse maps $T \to X$ and $T \to Y$.
- there exists a $\mathbb{Z}$-splitting T and equivariant edge folds $X \to T$, $Y \to T$.

Note that there is a natural inclusion $i : FS_n \to FZ_n$. If two free splittings are connected by an edge in FS_n, then their images are also connected by an edge of the first type in FZ_n. $Out(F_n)$ acts on FZ_n by simplicial automorphisms in the obvious way.

We can now extend this map i to a map f from the barycentric subdivision of FS_n to FZ_n as follows: A vertex V of FS_n' is a k-edge splitting of F_n. Define $f(V)$ to be the splitting obtained by collapsing all edges but one to a point. The map is only coarsely well-defined, but for any choice of edge in V, the one-edge splittings obtained will be at most distance 1 apart. Then extend to a graph map from $FS_n' \to FZ_n$.

Furthermore, f restricts to i on the vertices of FS_n (these are already one-edge splittings), and f is clearly 1-Lipschitz as well. We will need the following useful lemma.

Lemma 1.7. *Suppose X and Y are one-edge, two-vertex splittings connected by an edge of the second type in FZ_n. Then there exist vertices X' and Y' with $d(X, X') \leq 1$ and $d(Y, Y') \leq 1$ such that $d(X', Y') \leq 1$ and which share a vertex group.*

Proof. Let $\langle w \rangle$ be the edge group of the $\mathbb{Z}$-splitting to which X and Y fold, and let A be the smallest free factor containing $\langle w \rangle$ (in which case, we say that the element w *fills* A). Then there exist $X' = A * B$ and $Y' = A * B'$ such that X and X', and Y and Y' share a common refinement. $\square$

1.9 Hyperbolicity of FZ_n

We use the map $f : FS'_n \to FZ_n$ and the method pioneered in [29] to prove the main theorem:

Theorem 1.8. *The cyclic splitting graph FZ_n is δ-hyperbolic.*

Proof. Let S be the set of one-edge splittings in FS_n. Conditions (1) and (2) of Theorem 2.1 are clearly satisfied.

Since FS_n is δ-hyperbolic by [24], it suffices to show condition (3) is true: that there exists an $M > 0$ such that for any one-edge free splittings X and Y, if $d_{FZ_n}(f(X), f(Y)) \leq 1$, then $diam(f[X,Y]) \leq M$.

Suppose X and Y are one-edge free splittings of F_n which are joined by an edge of the second type in FZ_n (note that it suffices to cover this case because an edge of type 1 corresponds to being distance 1 in FS'_n). Suppose T is the $\mathbb{Z}$-splitting such that there exist edge folds $X \to T$ and $Y \to T$.

There are two cases to cover: (1) the graph of groups of the $\mathbb{Z}$-splitting T is a segment and (2) is a loop:

Case 1

By Lemma 5.2, choosing splittings at most distance 1 away we may assume $X = A * B$ and $Y = A * B'$, and that the element w fills A. Then the condition that X and Y fold to T is exactly the condition that $B * \langle w \rangle = B' * \langle w \rangle$. Let R and R' be free splittings defined as follows: both have underlying graphs which are roses, and the loops of R represent elements of bases of A and B; in particular choose a basis $\{a_1, \ldots, a_k\}$ of A and a basis $\{b_1, \ldots, b_l\}$ of B and label the edges of R by the collection of a_i's and b_j's. Denote the basis of F_n formed by this collection by β.

Choose a basis $\{b'_1, \ldots, b'_l\}$ for B' and define R' as the rose whose edges represent the elements in the basis $\beta' = \{a_1, \ldots, a_k, b'_1, \ldots b'_l\}$. Label the edges of R' by the elements of β' written in the basis β (subdividing the edges of R' as necessary) so that both are β-graphs. Note that we see a subgraph labelled by the a_i's in both. This gives a homotopy equivalence $R' \to R$. By perhaps conjugating, we may assume that the map $R' \to R$ is foldable; indeed $R' \to R$ fails to be foldable exactly when the word labeling each edge starts and ends with some a_i, so by conjugating every label we can be sure this does not happen without changing the splitting. Note that R (resp. R') has $d(f(R), X) \leq 1$ (resp. $d(f(R'), Y) \leq 1$).

Then choose a Handel-Mosher folding path $R' = \Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_N = R$ as follows: recall that $B * \langle w \rangle = B' * \langle w \rangle$ so that the basis elements $b'_1, \ldots, b'_j$ written in terms of β are just words in w and $b_1, \ldots, b_l$. Each maximal fold $\Gamma_i \to \Gamma_{i+1}$ occurs as one of the following two types, either (1) fold a loop labelled by some b'_j over a letter a_i in the word w or (2) fold maximal initial segments of two distinct loops labelled by some b_i and b_j. We require to fold an entire w before moving on to a fold of the second type: if we do a maximal fold of type 1 and fold only a proper subword of w, then after this fold we still see natural edges with the same initial label. By [24], regardless of the order in which the edges are folded, we still end up at R, so we continue doing maximal folds until we have folded out the entire word w.

In particular, each type of fold (1) or (2) either leaves the splitting $f(\Gamma_{i+1})$ (coarsely) equal to $f(\Gamma_i)$ or it gives another splitting $A*B_i$ within distance 1 of $f(\Gamma_i)$ so that $B_i*\langle w\rangle = B'*\langle w\rangle$. In either case, at each step of the folding path Γ_i, we have $d_{FZ_n}(f(\Gamma_i), f(X)) \leq 3$.

Case 2

Suppose the vertex group of X is $A*B$ and the vertex group of Y is $A*B'$, where A is the smallest free factor containing $\langle w\rangle$. X and Y are adjacent to a common $\mathbb{Z}$-splitting T exactly when $A*B*\langle w^t\rangle = A*B'*\langle w^t\rangle$, where t is the element of F_n corresponding to the nontrivial loop in the graph of groups, and $\langle w\rangle$ is the edge group of T.

We follow the same basic outline as in the segment case: choose splittings R and R' as follows: Let $\{a_1, \ldots, a_k\}$ be a basis of A, and $\{b_1, \ldots, b_l\}$ a basis of B. Let R be the splitting with underlying graph a rose and whose edges are labelled by the elements in the basis $\beta = \{a_1, \ldots, a_k, t, b_1, \ldots, b_l\}$. Choose a basis $\{b'_1, \ldots, b'_l\}$ for B' and let $\beta' = \{a_1, \ldots, a_k, t, b'_1, \ldots, b'_l\}$. We choose R' to be the rose whose edges represent the elements in the basis β'. Label the edges of R' by this elements of the basis β' written in the letters of β so that both R and R' become β-graphs.

By conjugating, we may assume that the homotopy equivalence $R' \to R$ given by the markings is foldable. Also recall that $A*B*\langle w^t\rangle = A*B'*\langle w^t\rangle$ so that every basis element b'_i is written as a word in $A*B*\langle w^t\rangle$. Thus, there is a Handel-Mosher folding path $R' \to \Gamma_0 \to \Gamma_1 \to \ldots \to \Gamma_N = R$ all of whose maximal folds $\Gamma_i \to \Gamma_{i+1}$ either (1) fold an edge labelled by some b'_i over a edge whose label is a letter in the word w^t or (2) folds an initial segment of an edge labelled by b_i with the initial segment of a different edge labelled by a_j or b_j. As above, with the first type of fold we make sure to fold an entire word w^t before moving on. In both cases, either the splitting $f(\Gamma_{i+1})$ is within distance 1 of $f(\Gamma_i)$ or $f(\Gamma_i)$ is distance 1 from a one-edge splitting whose vertex group is of the form $A*B_i*\langle w^t\rangle$.

In either case, for each i we have $d_{FZ_n}(f(\Gamma_i), f(X)) \leq 3$.

By [24], since each of the maps $\Gamma_i \to \Gamma_{i+1}$ are maximal folds, $d(\Gamma_i, \Gamma_{i+1}) \leq 2$. Since the map f is Lipschitz, this implies that the image of this folding path is contained in a bounded neighborhood of the splitting T. Furthermore, since folding paths are unparametrized quasi-geodesics by Theorem 4.1, this path is uniformly close to $[R', R]$, and because FS'_n is hyperbolic, any geodesic $[X, Y]$ is uniformly close to $[R', R]$. Putting all this together, we see that there exists a constant $M \geq 0$ such that $diam(f[X, Y]) \leq M$. □

1.10 Quasi-isometric Models of FZ_n

There are a few other candidate graphs that we might have called the cyclic splitting graph. We will show that all these graphs are $Out(F_n)$-equivariantly quasi-isometric.

Define the graph $\overline{FZ}_n$ as follows: vertices of $\overline{FZ}_n$ are one-edge free or $\mathbb{Z}$-splittings of F_n. Two such vertices X,Y are connected if (1) X and Y are free splittings which admit a common refinement or (2) X can be obtained from Y by an edge fold.

Proposition 1.5. *FZ_n and $\overline{FZ}_n$ are $Out(F_n)$-equivariantly quasi-isometric.*

Proof. Define a map $\phi : FZ_n \to \overline{FZ}_n$ in the obvious way: send a vertex of FZ_n to the corresponding free splitting in $\overline{FZ}_n$. Extend to a map of the entire graph. ϕ is clearly equivariant.

Let $X, Y \in V(FZ_n)$ with $d(X, Y) \leq 1$. Then by definition of $\overline{FZ}_n$, at worst we have $d(\phi(X), \phi(Y)) \leq 2$. Hence $d(X, Y) \leq 2d(\phi(X), \phi(Y))$. Furthermore, if $d(f(X), f(Y)) \leq 2$, then X and Y are joined by a path of length at most 2. In particular,

$$\frac{1}{2}d(X, Y) \leq d(\phi(X), \phi(Y)) \leq 2d(X, Y)$$

so ϕ is quasi-isometry as desired. □

There is third graph whose definition more closely resembles the definition of FS_n. Define a graph C_n whose vertices are one-edge free or $\mathbb{Z}$-splittings and where two vertices X and Y are connected by an edge if the corresponding splittings have a two-edge common refinement.

Proposition 1.6. *FZ_n and C_n are $Out(F_n)$-equivariantly quasi-isometric.*

Proof. We will actually show that there is an $Out(F_n)$-equivariant quasi-isometry ϕ : $\overline{FZ}_n \to C_n$. The vertex sets of $\overline{FZ}_n$ and C_n are the same, so set ϕ to be the identity on vertices. Then extend ϕ to a map of graphs.

Let X and Y be vertices of $\overline{FZ}_n$ such that $d(X, Y) \leq 1$. Then by folding the edge group $\langle w \rangle$ "half-way" over the edge of the splitting, we get a two-edge splitting which commonly refines both X and Y. More precisely, if X and Y are one-vertex splittings, consider the two-edge splitting in which the edges are adjacent at both endpoints, the vertex groups are the vertex group of X and the edge group $\langle w \rangle$ of the $\mathbb{Z}$-splitting, and the edge groups are trivial and $\langle w \rangle$.

Similarly, if X and Y are two-vertex splittings, say $X = A * B$ and $Y = A *_{\langle w \rangle} \langle B, w \rangle$, then we can consider the two-edge splitting $A *_{\langle w \rangle} \langle w \rangle * B$ which refines X and Y. Hence $d(\phi(X), \phi(X)) \leq d(X, Y)$.

Now suppose X and Y are cyclic splittings which are commonly refined by a two-edge splitting (so $d(\phi(X), \phi(Y)) \leq 1$). We need to find a uniform $L > 0$ such that $d(X, Y) \leq L$.

Suppose that X is a $\mathbb{Z}$-splitting and Y is a free splitting. Then we can unfold the edge group from X to get a free splitting X'. Since X and Y are commonly refined, writing down the vertex groups of X' we see that X' is commonly refined with Y, and hence $d(X, Y) \leq 2$ in $\overline{FZ}_n$. We will do one case carefully - the others are similar and left to the reader. Suppose the graphs of groups corresponding to both X and Y are segments. Then the common refinement is $A *_{\langle w \rangle} B * C$, with $X = A *_{\langle w \rangle} (B * C)$ and $Y = (A *_{\langle w \rangle} B) * C$.

Suppose furthermore that $A *_{\langle w \rangle} (B * C)$ unfolds to $X' = A * (B' * C)$. Then X' and Y are commonly refined by the two-edge splitting $A * B' * C$ since the factors $A *_{\langle w \rangle} B$ and $A * B'$ are equal. The other cases are similar, so we have the above inequality for $L = 2$.

Suppose both X and Y are $\mathbb{Z}$-splittings which are commonly refined by a two-edge splitting, say $A *_{\langle s \rangle} B *_{\langle t \rangle} C$, so in particular $X = (A *_{\langle s \rangle} B) *_{\langle t \rangle} C$ and $Y = A *_{\langle s \rangle} (B *_{\langle t \rangle} C)$. By Theorem 5.1 above, one of the edge groups $\langle s \rangle$ or $\langle t \rangle$ can be unfolded to get a splitting with one trivial edge stabilizer, and one $\mathbb{Z}$-stabilizer. We can then unfold the remaining $\mathbb{Z}$-edge to get a free splitting, say $A' * B' * C'$ which is a common refinement of the one-edge free splittings $(A' * B') * C'$ and $A' * (B' * C')$. Furthermore, we can fold the element t over the edge of the splitting $(A' * B') * C'$ to get a $\mathbb{Z}$-splitting equal to X and folding s over the edge of $A' * (B' * C')$ we obtain the splitting Y.

Hence, $d(X, Y) \leq 3$. Therefore, we have

$$\frac{1}{3} d(X, Y) \leq d(\phi(X), \phi(Y)) \leq 3 d(X, Y)$$

so $\phi : \overline{FZ}_n \to C_n$ is a quasi-isometry. □

1.11 FS_n and FZ_n Are Not Equivariantly Quasi-isometric

There are natural $Out(F_n)$-equivariant maps: $FS_n \to FZ_n$ discussed above and $FZ_n \to FF_n$, which is given by sending a vertex of FZ_n to one of the edge groups of the corresponding free splittings. There is also a natural map $FS_n \to FF_n$ defined in the same way, which clearly factors through the above maps. It is a priori unclear that these maps are not quasi-isometries.

Proposition 1.7. *For $n \geq 3$, the natural $Out(F_n)$-equivariant map $FS_n \to FZ_n$ is not a quasi-isometry.*

Note that any two equivariant, coarsely Lipschitz maps are bounded distance, so Proposition 8.1 implies that there does not exist any equivariant quasi-isometry $FS_n \to FZ_n$.

Proof. We assume the following result claimed in [24], and which is to appear in another part of their work on the free splitting complex: An element $\phi \in Out(F_n)$ acts hyperbolically on FS_n is there exists an attracting lamination Λ of ϕ whose support is all of F_n, i.e., Λ is not carried by a proper free factor.

Let S be a surface so that $\pi_1(S)$ is free of rank $n \geq 3$, and suppose there exists a (nonseparating) simple closed curve C in S such that $S \setminus C = \Sigma$ admits a pseudo-Anasov mapping class. C gives a $\mathbb{Z}$-splitting $\pi_1(S) = A*_C$. Such a surface exists for all $n \geq 3$: indeed, for $n \geq 4$, we can take a surface of genus ≥ 2 with either one or two boundary components. To get $n = 3$, take a twice-punctured torus. We can cut along a curve C to get a 4-times punctured sphere, which admits a pseudo-Anasov homeomorphism.

Let ϕ be a pseudo-Anasov mapping class of Σ. Hence ϕ can also be thought of as an outer automorphism of $\pi_1(S)$ which fixes the splitting $A*_C$. It remains to show that the expanding lamination Λ of ϕ is not carried by a proper free factor. Since ϕ is pseudo-Anasov, it follows that Λ restricted to Σ is minimal and filling.

Suppose not, and let H be a factor which carries a leaf L of Λ (and hence carries all of Λ by minimality). Let $\tilde{S}$ be the cover of S corresponding to H. Since H is finitely generated, there exists a compact subsurface S_H of $\tilde{S}$ which has fundamental group H. Let $\tilde{L}$ be a lift of L to $\tilde{S}$ which is contained in S_H. Let Σ' be the smallest subsurface of S_H which $\tilde{L}$ fills. Suppose $\tilde{L}_0$ is any other lift of L which meets Σ'. If $\tilde{L}_0$ is not contained in Σ', it enters through some boundary component. By Theorem 5.2 in [18] (this is for foliations, but the result for laminations is similar), $\tilde{L}_0$ exits through a boundary component of Σ'. In particular, by cutting Σ' along $\tilde{L}_0$ we would obtain a smaller surface which $\tilde{L}$ fills, a contradiction. Hence, any lift of L meeting Σ' must be contained in Σ'.

In particular, Σ' is (homotopic to) a finite cover of Σ, so H contains a finite index subgroup of $\pi_1(\Sigma) = A$, which is impossible unless $H = F_n$. Indeed, $\pi_1(\Sigma)$ cannot be contained in a proper free factor; looking at Euler characteristics we see $\chi(S) = \chi(\Sigma)$ so $\chi(\tilde{\Sigma}) \leq \chi(S)$, so the rank of H is at least n.

$\square$

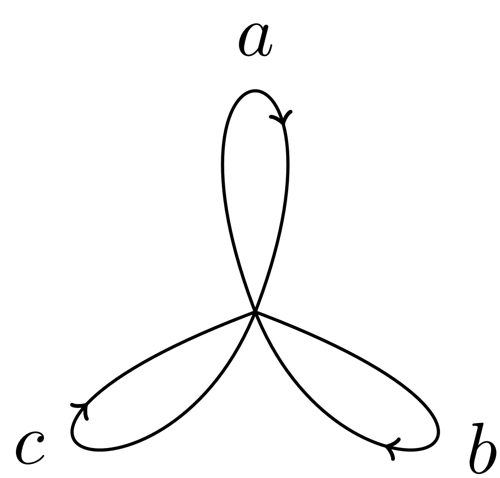

Figure 1.1. The free group on 3 generators $F_3 = \langle a, b, c \rangle$ as the fundamental group of a rose.

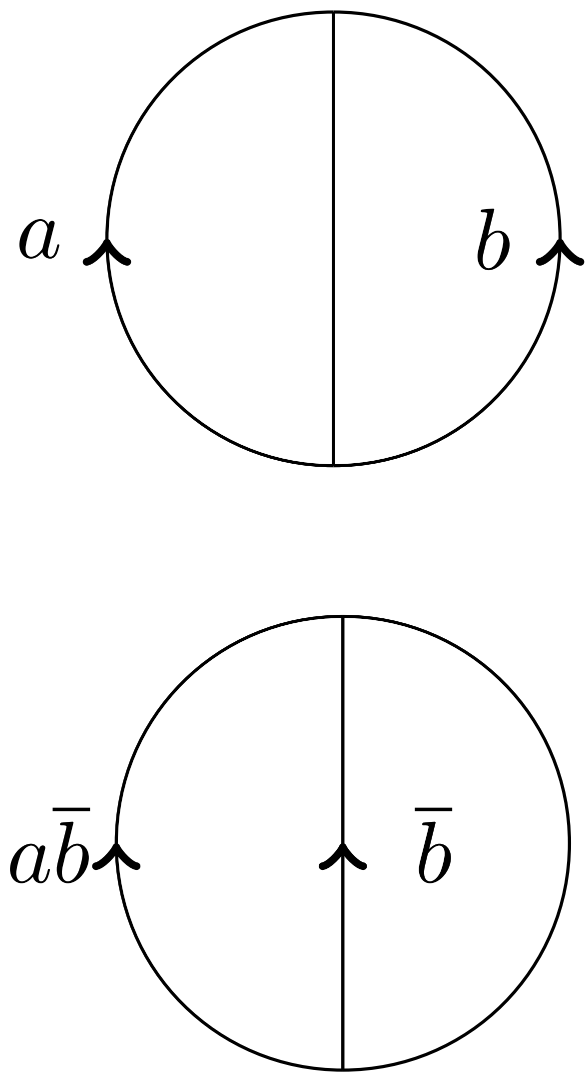

Figure 1.2. Equivalent markings of the theta graph

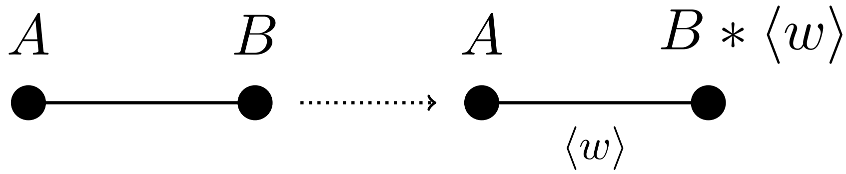

Figure 1.3. Folding a two-vertex splitting

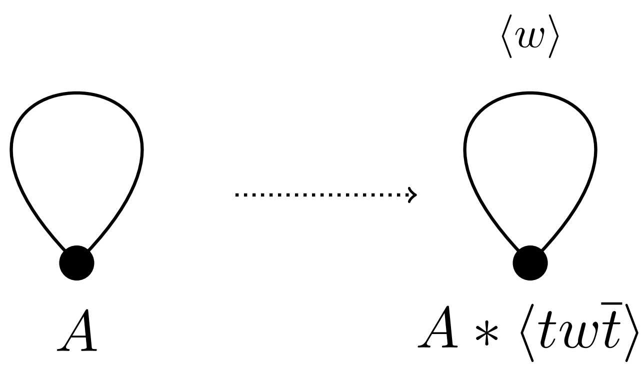

Figure 1.4. Folding a one-vertex splitting. The nontrivial loop represents the element t.

CHAPTER 2

WEAK PROPER DISCONTINUITY

In [7], Bestvina and Feighn prove that the action of $Out(F_N)$ on the free factor graph satisfies the Weak Proper Discontinuity condition of Bestvina and Fujiwara [9]. We give an alternative proof using the North-South dynamics of the action of fully irreducible outer automorphims on CV_N and the space of currents.

2.1 Trees and Currents

We start this section with some brief background on trees and currents. For more details see, for example, [27]. An $\mathbb{R}$-tree T with a minimal action of F_N is said to be *very small* if

1. Arc stabilizers are either trivial or maximal cyclic.

2. Stabilizers of nondegenerate tripods are trivial.

For $g \in F_N$, let $l_T(g) = \min_{x \in T} d(x, gx)$ denote the *translation length of g in T*. Note that $l_T(g^m) = m l_T(g)$. The function l_T is clearly invariant under conjugation in F_N.

Let cv_N denote the space of minimal, free, simplicial F_N actions on $\mathbb{R}$-trees, modulo the relation that $T \sim T'$ if there is an F_N-equivariant isometry $T \to T'$. We call cv_N *unprojectivized outer space*. Define $\bar{\mathrm{cv}}_N$ to be the space of all very small actions of F_N on $\mathbb{R}$-trees. $\bar{\mathrm{cv}}_N$ is identified with the length-function compactification of cv_N.

Furthermore, another definition of CV_N is to take cv_N and projectivize: that is, take CV_N to be all projective classes of minimal, free, simplicial F_N actions on $\mathbb{R}$-trees, and define $\overline{CV_N}$ to be projective classes of very small actions of F_N on $\mathbb{R}$-trees. Again, $\overline{CV_N}$ is naturally the compactification of CV_N [5].

We adopt a natural *left* action of $Out(F_N)$ on these spaces, where $\phi T = T\phi^{-1}$. In particular, $l_{\phi T}(g) = l_T(\phi^{-1} g)$

F_N with the word metric is hyperbolic. Let ∂F_N denote its Gromov boundary. Define $\partial^2 F_n := (\partial F_N \times \partial F_N \setminus \Delta)/\mathbb{Z}^2$, where the $\mathbb{Z}^2$ action implied is the natural "flip" action. A *geodesic current* on F_N is a positive Radon measure on $\partial^2 F_N$. Let $Curr_N$ denote the set of all currents on F_N. There is a natural *left* action of $Out(F_N)$ on $Curr_N$ by

$$\phi\nu(A) = \nu(\phi^{-1}A)$$

If $g \in F_N$, let $[g]$ denote the conjugacy class of g. Let $g^\infty = \lim_{n\to\infty} g^n$ and $g^{-\infty} = \lim_{n\to-\infty} g^n$. So given an element $g \in F_N$, one obtains a point $(g^\infty, g^{-\infty}) \in \partial^2 F_N$.

Suppose g is nontrivial and not a proper power. Let A_g be the collection of all F_N translates of $(g^\infty, g^{-\infty})$ and $(g^{-\infty}, g^\infty)$. Define

$$\mu_g = \sum_{(x,y)\in A_g} \delta_{(x,y)}$$

where $\delta_{(x,y)}$ is the dirac measure supported on the point $(x, y) \in \partial^2 F_N$.

For $g = f^m$, define $\mu_g = m\mu_f$. If $\nu \in Curr_N$ is a multiple of a current of the form μ_g, we say ν is a *rational current*.

Theorem 2.1 (Kapovich)**.** *The set of rational currents is dense in* $Curr_N$*.*

Analogous to the idea of an intersection number for curves on surfaces, there is a continuous, $Out(F_N)$-equivariant "intersection form"

$$\langle\ ,\ \rangle : \bar{\mathrm{cv}}_N \times Curr_N \to \mathbb{R}_{\geq 0}$$

This function is homogeneous in the first coordinate, linear in the second, and has the following values on rational currents:

$$\langle T, \mu_g\rangle = l_T(g)$$

Note that such a function must be unique.

Let $\mathbf{P}Curr_N$ denote the space of projective classes of currents. Define the *min-set* $M_N \subseteq \mathbf{P}Curr_N$ to be the closure of the set of classes of rational currents corresponding to primitive conjugacy classes. Fully irreducible elements in $Out(F_N)$ act with north-south dynamics on $\overline{CV_N}$ [30] and M_N [31].

Note that we will often use a conjugacy class and the current to which it corresponds interchangably.

2.2 The Free Factor Graph

Here we define a different (quasi-isometric) model of the 1-skeleton of the free factor complex.

Definition 2.1 (Free Factor Graph)**.** *The* free factor graph $\mathcal{FF}_N$ *is a graph whose vertex set is conjugacy classes of primitive elements in* F_N*. Two such vertices* a *and* b *share an edge if there exists a very small* F_N*-tree* T *such that* $l_T(a) = l_T(b) = 0$ *(i.e., both* a *and* b *are elliptic in some common tree* T*).*

Proposition 2.1. $\mathcal{FF}_N$ *is quasi-isometric to the 1-skeleton of the standard free factor complex* FF_N*.*

Proof. Let $f : \mathcal{FF}_N \to FF_N$ be the map which sends $a \mapsto \langle a \rangle$.Then f is Lipschitz; indeed, if a and b are elliptic in a common tree, then by [10], their images are some universally bounded distance in FF_N.

Furthermore, if $d_{FF_N}(\langle a \rangle, \langle b \rangle) = 2$, then there is some factor H containing both. Hence, any tree in which H stabilizes a vertex has both a and b elliptic. Since any two rank 1 factors are connected by a geodesic path which is a concatenation of length 2 paths between rank 1 factors, the result follows. □

2.3 Weak Proper Discontinuity

The notion of Weak Proper Discontinuity is discussed in more detail in [9]. We give a brief definition. Suppose X is a hyperbolic metric space. An action of G on X satisfies *Weak Proper Discontinuity (WPD)* if

1. G is not virtually cyclic
2. At least one element of G acts as a hyperbolic isometry
3. For every hyperbolic $g \in G$, every $x \in X$ and $C > 0$, there exists an $N > 0$ so that the set $\{\gamma \in G | d(x, \gamma x) \leq C, d(g^N x, \gamma(g^N x)) \leq C\}$ is finite.

First, we deduce a useful lemma from work of Bestvina-Feighn and Guirardel.

Lemma 2.2. *Suppose* a *and* b *are conjugacy classes which do not both fix a point in any common minimal, very small, simplicial* F_N*-tree. Then the intersection of the stabilizers of* a *and* b *in* $Out(F_N)$ *is finite.*

Proof. Suppose not. Let g be an infinite order element in the intersection of the stabilizers of a and b. Then a stable tree $T = \lim g^n T_0$, for $T_0 \in \overline{CV_N}$, has both a and b elliptic.

By [21] and [5], T can be approximated by an action of F_N on a very small simplicial tree in which a and b are elliptic. This is a contradiction. □

Armed with Lemma 2.2, we can prove

Theorem 2.3. *The action of $Out(F_N)$ on FF_N satisfies WPD.*

Proof. Conditions (1) and (2) are satisfied. Let f be a fully irreducible automorphism. Let L be the uniform bound on the diameter in FF_N of the set of reducing factors for a very small F_N-tree given in [10]. Let a and b be two conjugacy classes on a quasi-axis for f which are far apart.

By way of contradiction, suppose there exists an infinite collection of $g_n \in Out(F_N)$ such that $d(a, g_n a) \leq C$ and $d(b, g_n b) \leq C$ for all n. By north-south dynamics of the action of $Out(F_N)$ on $\overline{CV_N}$ and M_N (the closure of the set of primitive currents in $\mathbf{P}Curr_N$) (see [31]), there exists attracting and repelling trees and currents T_+, T_-, μ_+, μ_- so that $\langle T_+, \mu_+\rangle = 0$ and $\langle T_-, \mu_-\rangle = 0$.

By continuity of the intersection pairing, there exists closed neighborhoods $U_0^{\pm} \subset \cdots \subset U_{C+1}^{\pm}$ of $T_{\pm}$ and $V_1^{\pm} \subset \cdots \subset V_{C+1}^{\pm}$ of $\mu_{\pm}$ so that the following holds:

- If $T \in U_i^{\pm}$ and $\langle T, \mu\rangle = 0$, then $\mu \in V_{i+1}^{\pm}$
- If $\mu \in V_i^{\pm}$ and $\langle T, \mu\rangle = 0$, then $T \in U_i^{\pm}$
- If $T \in U_{C+1}^{\pm}$ and $\mu \in V_{C+1}^{\mp}$, then $\langle T, \mu\rangle \neq 0$.

We may assume that $a \in V_1^+$ and $b \in V_1^-$. Note that, by definition of the free factor graph, if c is connected by an edge to a, then $c \in V_2^+$, and by induction, if $d(a, c) \leq C$, then $c \in V_{C+1}^+$. The same holds for the neighborhoods around b.

We conclude that, up to subsequence, $g_n(a) \to A \in V_{C+1}^+$ and $g_n(b) \to B \in V_{C+1}^-$.

There are two cases to consider: (1) if the currents $g_n(a)$ are all different or (2) $g_n(a)$ and $g_n(b)$ take finitely many values. Case (2) is easier, so we cover that one first.

Passing to a subsequence, we may assume $g_n(a)$ and $g_n(b)$ are constant sequences. Hence, the intersection of the stabilizers of a and b is infinite. But Lemma 1.2 ensures that this is absurd.

For case (1), we have that (up to subsequence, of course) all $g_n(a)$ are distinct. To get convergence in the space of *unprojectivized* currents, one must rescale by some constants $\lambda_n > 0$ so that $\frac{g_n(a)}{\lambda_n} \to \hat{A}$, where λ_n tends to ∞ as $n \to \infty$. Let T be a tree with $\langle T, b\rangle = 0$, so $T \in U_{C+1}^-$. Up to subsequence, $g_n T \to \hat{T} \in U_{C+1}^-$, and $\langle \hat{T}, b\rangle = 0$. We have $\langle \hat{T}, \hat{A}\rangle = \lim\langle \hat{T}, \frac{g_n(a)}{\lambda_n},\rangle = \lim \frac{1}{\lambda_n}\langle \hat{T}, g_n(a)\rangle = 0$, so $\langle \hat{T}, A\rangle = 0$ contradicting the last bullet above.

□

2.4 WPD for the Intersection Graph

A conjugacy class in F_N is called *geometric* if it is either primitive or represented by the boundary of a marked surface.

We define the following two graphs:

Definition 2.2. *The* intersection graph*, I_N, is a bipartite graph with vertex set consisting of very small, minimal, simplicial, nonfree F_N-trees and rational currents corresponding to geometric conjugacy classes. A tree T and a current μ_a are connected by an edge if $\langle T, \mu_a \rangle = 0$ (i.e., the conjugacy class a has length 0 in the tree T).*

Definition 2.3. *The graph P_N has a vertex set consisting of marked roses (up to the standard equivalence). Two roses are connected by an edge if either (1) R and R' share an edge with the same label or (2) there exists a marked surface with 1 boundary component, such that w is the element in $\pi_1(S)$ represented by the boundary, and w crosses each edge of R and R' twice.*

Proposition 2.2. *P_N is hyperbolic.*

Proof. Again, we adopt the Kapovich-Rafi machinery [29]. There is an obvious Lipschitz, surjective map $f : FS_N \to P_N$. What we need to show is that, if $d(R_1, R_2) = 1$ in P_N, then the folding path between them stays a uniformly bounded distance from R_1.

Note that if R_1 and R_2 share a common basis element, then this is proved in Kapovich-Rafi, so it suffices to assume that there exists a surface boundary word w which crosses each edge of R_1 and R_2 twice. To show the claim, it remains to show that in a folding path from R_1 to R_2, the length of the word w stays a uniformly bounded in each graph.

Note that the number of illegal turns does not increase along the folding path. Let K be the number of illegal turns, and n be the number of illegal turns crossed by the loop representing w. If $l(w) > 4$ at any point along the folding path, then folding increases the length of the loop representing w. Indeed, if we fold each illegal turn by ϵ, then the length of w becomes $l(w) - 2\epsilon n$. However, to maintain volume 1, we must rescale by $\frac{1}{1-\epsilon K}$. It follows that if $l(w) > 4$, then in the folded graph, the loop representing w is strictly longer.

Hence, if w becomes too long at any point in the folding path, it continues to grow. But in R_2, the length of w is 2 by hypothesis. So the length of w along any such folding path is uniformly bounded (in fact, less than 4). □

Proposition 2.3. *P_N and I_N are quasi-isometric.*

Before we prove the result, we recall a helpful theorem of Skora:

Theorem 2.4 (Skora [39]). *Suppose S is a hyperbolic surface. Let T be a small $\pi_1(S)$-tree such that each cusp subgroup fixes a point in T. Then T is dual to a measured lamination on S.*

Proof of Proposition. Define a map $f : P_N \to I_N$ as follows: given a rose R, let $f(R)$ be the one-edge $\mathbb{Z}$-splitting obtained by collapsing all edges of R but one to a point, and then folding a primitive element w in the vertex group across the edge.

Clearly, this map is coarsely surjective. Suppose R_1 and R_2 are vertices in P_N with $d(R_1, R_2) = 1$. If R_1 and R_2 share a basis element, then $d(f(R_1), f(R_2)) \leq 4$.

If w is a surface boundary word which crosses each edge of R_1 and R_2 twice, then there exists edge collapses and folds so that the resulting $\mathbb{Z}$-splittings have w elliptic. These may not have been the points $f(R_1)$ and $f(R_2)$ we originally choose, but in any case, $d(f(R_1), f(R_2)) \leq 6$. So the map f is Lipschitz.

Suppose now that there exist two roses R_1 and R_2 so that $d(f(R_1), f(R_2)) \leq 1$. If $f(R_1)$ and $f(R_2)$ have a common primitive elliptic, then $d(R_1, R_2)$ are uniformly bounded by [29]. So suppose the common elliptic is a surface boundary word w. Let S be the marked surface whose boundary is w.

By Skora's theorem, $f(R_1)$ and $f(R_2)$ are dual to curves C_1 and C_2 on S. Connecting these curves to the boundary gives unfoldings of $f(R_1)$ and $f(R_2)$ to free splittings T_1 and T_2. These are probably not the free splittings obtained in the construction of $f(R_i)$ by collapsing all edges of the R_i but one to a point. However, note that any two unfoldings of a $\mathbb{Z}$-splitting are uniformly bounded distance in the free factor complex. Indeed, the vertex groups of any two unfoldings fix a point in the $\mathbb{Z}$-splitting. Hence by [10], the vertex groups are uniformly bounded in the factor complex. Therefore, by [29], the corresponding roses obtained by "blowing-up" the vertex groups are uniformly bounded in P_N.

Further, the unfolded splittings T_1 and T_2 are both dual to arcs on the same surface. These arcs can be completed to arc systems which are dual to roses R_1' and R_2' so that w crosses each edge of both twice. By the above, it follows that R_1 and R_2 are uniformly bounded distance. Hence, the map f is a quasi-isometry. □

Theorem 2.5. *Atorodial, fully irreducible automorphisms act hyperbolically on I_N, and the action of $Out(F_N)$ on I_N satisfies WPD.*

Proof. By [28], atorodial iwips act on I_N with unbounded orbits. Let f be such an automorphism. Choose a and b geometric conjugacy classes which are far apart in an orbit of f.

In order to apply the proof of Theorem 1.1 to Theorem 2.2, we will need one more quasi-isometric model of the intersection graph. Let G_N be the graph whose vertex set consists of geometric conjugacy classes, and where two such are connected by an edge there exists a minimal, simplicial, very small tree in which both are elliptic. There is a natural map $h : G \to I_N$ which sends any vertex to the corresponding current, and which sends any edge to a geodesic path joining the images of the terminal vertices. Such a map is clearly 2-Lipschitz. Furthermore, if $h(x)$ and $h(y)$ are within distance 2, then this means that the corresponding conjugacy classes are elliptic in a common tree. Hence x and y are joined by an edge in G, so h is a quasi-isometry.

By replacing the free factor graph with G, and the minset M_N by the entire current space $\mathbf{P}Curr_N$, the entire proof of Theorem 1.1 applies to this Theorem 2.2.

The argument further implies that the orbit of any point by f^n makes linear progress in n, so f acts hyperbolically on I_N.

$\square$

CHAPTER 3

CONSTRUCTING NONUNIQUELY ERGODIC ARATIONAL TREES

3.1 Introduction[2]

Let $F = F_N$ denote the rank-N free group, with $N \geq 4$. A *factor* is a conjugacy class of nontrivial, proper free factors of F_N; a conjugacy class of elements of F_N is *primitive* if any of its representatives generates a representative of a factor. A *curve* is a one-edge $\mathbb{Z}$-splitting of F_N with primitive edge group; every curve is a very small tree. Two curves T, T' are called *disjoint* if there is a two-edge simplicial tree Y such that both T and T' can be obtained by collapsing the components of an orbit of edges in Y to points; this is the same as having that $l_T + l_{T'}$ is a length function for a very small tree, namely Y.

A *measure* on a tree T is a collection of finite Borel measures $\{\mu_I\}$, where I runs over finite arcs of T, that is invariant under the F_N-action and is compatible with restriction to subintervals; if T has dense orbits, then the set of measures on T is a finite dimensional convex cone [22]. The measure of a finite subtree $K \subseteq T$ is the sum of measures of the arcs in any partition of K into finitely many arcs. The *co-volume* of T is the infimum of measures of a finite forests K such that $T \subseteq \cup_{g \in F_N} gK$. If $H \leq F_N$ is finitely generated and does not fix a point of T, then there is a unique minimal H invariant subtree $T_H \subseteq T$; define the co-volume of H, denoted $covol(H)$, to be the co-volume of T_H, for T fixed. A tree T is called *arational* if $covol(F') > 0$ for every factor F'; this is the same as having every factor act freely and simplicially on some invariant subtree of T; see [10, 36]. An arational tree T is called *nonuniquely ergodic* if there are nonhomothetic measures μ and μ' for T; see [14, 22, 35] for more about measures on trees. Our main result is:

Theorem: Let T, T' be disjoint curves with neighborhoods U, U'. There is a 1-simplex of nonuniquely ergodic, arational, nongeometric trees with one endpoint in each of U, U'.

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[2]Most of this chapter appears as a preprint at http://arxiv.org/abs/1311.1771. This is joint work with Patrick Reynolds.

Notions around geometric trees are reviewed in Section 3.2. Examples of nonuniquely ergodic arational trees dual to measured foliations on surfaces are well-known, but all such trees are geometric; R. Martin contructed one example of a nonuniquely ergodic tree that is geometric and of *Levitt type* [32]. Our procedure gives the first examples of nonuniquely ergodic, arational trees that are nongeometric.

3.1.1 Analogy with Gabai's Construction and Outline of Proof

Our proof of the Theorem is an adaptation of an idea of Gabai, and there are two main technical steps; before explaining those, we recall the proof of the following:

Theorem 3.1. *[19, Theorem 9.1] Let Σ be a k-simplex of disjoint curves in $\mathcal{PML}$, and $U_1, \dots, U_{k+1}$ be neighborhoods of the extreme points. There is a k-simplex Σ' of nonuniquely ergodic minimal and filling laminations with extreme points in the U_j's.*

Theorem 3.1 follows at once by induction after showing:

Proposition 3.1. *[19] Let $\alpha_1, \dots, \alpha_r \in \mathcal{PML}(S_{g,p})$ be a collection of disjoint curves; let U_j be a neighborhood of α_j, and let c be a curve. There are disjoint curves $\alpha_1', \dots, \alpha_r'$ so that $\alpha_j' \in U_j$ with neighborhoods $U_j' \subseteq U_j$, such that $i(\beta, c) > d_c > 0$ for any $\beta \in \cup_j U_j'$.*

Here is how Gabai proves Proposition 3.1. The surface S is cut into pieces $P_1, \dots, P_k$ by the α_j's, and glueing back along a fixed α_j gives a surface σ that is at least as complex as a 4-punctured sphere or a punctured torus; α_j is essential and nonperipheral on σ. A generic choice γ of a curve in σ intersects α_j and all arcs of c cutting σ (to be more concrete, one can apply a high power of a pseudo-Anosov on σ to α_j to get γ); now apply a high power of a Dehn twist in α_j to γ to get a curve α_j' in U_j, and replace α_j with α_j'. To ensure positive intersection of every α_j' with c, one begins by modifying α_j, chosen so that either c meets σ in an essential arc or else $c = \alpha_j$; after finding α_j', move to a boundary component $\alpha_{j'}$ of σ and continue. Since $i(c, \alpha_j') > 0$, continuity of $i(\cdot, \cdot)$ ensures that we can find neighborhoods $U_j' \subseteq U_j$ such that $i(c, \beta) > 0$ for any $\beta \in \cup_j U_j'$.

We proceed essentially in the same way, with our "curves" serving as surrogates of curves on a surface. The analogue, for a tree T, of having positive intersection with every simple closed curve on a surface is that every factor acts with positive co-volume in T (this is the same as every factor acting freely and simplicially on its minimal subtree of T [36]), and our analogue of continuity of $i(\cdot, \cdot)$ is the continuity of the Kapovich-Lustig intersection function combined with continuity of the restriction map from the space of very small F_N-trees to the space of H-trees for H a finitely generated subgroup of F_N. Since our aim is to construct limiting trees that are both arational and nongeometric, we have two versions of Proposition 3.1; these are the two main technical steps in our argument and appear as Propositions 3.3 and 3.4 below.

This paper is organized as follows: In Section 3.2, we give relevant background about trees, currents, and laminations. In Section 3.3, we give our analogue of Proposition 3.1 that ensures arational limiting trees, while in Section 3.4, we give our analogue of Proposition 3.1 that ensures nongeometric limiting trees. Section 3.5 contains the proof of the main result.

3.2 Background

Fix a basis $\mathscr{B}$ for F_N. Use $\bar{\mathrm{cv}}_N$ to denote the set of very small F_N-trees; cv_N denotes the subset of free and simplicial very small trees; and $\partial\mathrm{cv}_N = \bar{\mathrm{cv}}_N \setminus \mathrm{cv}_N$ [5, 13]. If F' is another finite rank free group, we use $cv(F')$ and $\partial cv(F')$ to denote the corresponding spaces of F'-trees. If T is a very small tree, then l_T denotes its length function; spaces of trees get the length functions topology, which coincides with the equivariant Gromov-Hausdorff topology.

3.2.1 Currents and Laminations

Use ∂F_N to denote the boundary of some $T_0 \in \mathrm{cv}_N$, and put $\partial^2 F_N := (\partial F_N \times \partial F_N \setminus diag.)/\mathbb{Z}_2$; this can be thought of as the space of unoriented geodesic lines in T_0. The obvious action of F_N on ∂F_N gives an action of F_N on $\partial^2 F_N$. A *lamination* is a nonempty, invariant, closed subset $L \subseteq \partial^2 F_N$. A *current* is a nonzero invariant Radon measure ν on $\partial^2 F_N$; the support of a current ν, denoted $Supp(\nu)$, is a lamination; the set of currents gets the weak-* topology and is denoted by $Curr_N$. There is a continuous function, called *intersection*,

$$\langle \cdot, \cdot \rangle : \bar{\mathrm{cv}}_N \times Curr_N \to \mathbb{R}_{\geq 0}$$

that is homogeneous in the first coordinate and linear in the second coordinate [27].

If $T \in \partial\mathrm{cv}_N$, then either T is not free or else T is not simplicial; hence for any $\epsilon > 0$, there is $g \in F_N$ with $l_T(g) < \epsilon$. Define

$$L(T) := \cap_{\epsilon>0} \overline{\{(g^{-\infty}, g^{\infty}) | l_T(g) < \epsilon\}}$$

The set $L(T)$ is a lamination [15]. Kapovich and Lustig gave a complete characterization of when a tree and a current have intersection equal to zero: $\langle T, \nu \rangle = 0$ if and only if $Supp(\nu) \subseteq L(T)$ [28].

3.2.2 Geometric Trees

The topology on $\partial \mathrm{cv}_N$ is metrizable, and we fix a compatible metric. We record a lemma that follows immediately from the definition of the Gromov-Hausdorff topology.

Lemma 3.2. *For any finitely generated $H \leq F_N$, the function*

$$cv_N \to cv(H) : T \mapsto T_H$$

is continuous.

Considering the Gromov-Hausdorff topology, one gets a function $\mathrm{cv}_N \ni T \mapsto x \in T$ that is "continuous" in the following sense: given a finite subset $S \subseteq F_N$ and $\epsilon > 0$, there is a $\delta > 0$ so that if T' is δ-close to T, then the partial action of S on the convex hull of Sx_T is ϵ-approximated by the partial action of S on the convex hull of $Sx_{T'}$; this point is explained in [38]; in particular, see Skora's discussion of Proposition 5.2. We call this function a *continuous choice of basepoint* on cv_N.

We very quickly recall some notions around geometric trees; see [6] for details. Every $T \in \partial \mathrm{cv}_N$ admits *resolutions* as follows: fix $x \in T$ and let B_n be the n-ball in the Cayley tree for F_N with respect to $\mathscr{B}$. The partial isometries induced by elements of $\mathscr{B}$ on the convex hull $K(T, x_T, n)$ of $B_n x$ form a pseudo-group, which can be suspended to get a band complex $Y = Y(T, x, n)$, which is dual to a very small tree T_n, and $T_n \to T$ as $n \to \infty$. The geometric trees T_n come with morphisms $f_n : T_n \to T$, and T is geometric if and only if f_n is an isometry for $n >> 0$. The band complex Y decomposes via Imanishi's theorem into a union of finitely many maximal families of parallel compact leaves, called *families*, and finitely many minimal components, which are glued together along singular leaves; each family $\mathcal{C}$ has a well-defined *width*, denoted $w(\mathcal{C})$; see also [20]. The family $\mathcal{C}$ is called a *pseudo-annulus* if every leaf contains an embedded copy of S^1; the family $\mathcal{C}$ is called *nonannular* if it is not a pseudo-annulus.

3.2.3 Dehn Twists

Let T be a curve; for simplicity, assume T/F_N is a circle. Choosing an edge e in T gives F_N the structure of an HNN-extension $F_N = \langle a_1, \ldots, a_{N-1}, t, w' | w' = w^t \rangle$, where $w \in \langle a_1, \ldots, a_{N-1} \rangle$. The subgroup $V = \langle a_1, \ldots, a_{N-1}, w' \rangle$ is the stabilizer of one of the endpoints of e, and the subgroup $\langle w \rangle$ is the stabilizer of e. The element t is called the stable letter for this HNN-structure. Given this choice of e, one gets a *Dehn twist automorphism* τ of F_N, defined by $\tau(t) = tw$ and $\tau(a_i) = a_i$; the element w is called the *twistor*. The class of τ in $Out(F_N)$ does not depend on the choice of e and also is called a Dehn twist automorphism. Cohen and Lustig prove the following; see [13, Theorem 13.2].

Proposition 3.2. *[13] Let τ be a Dehn twist with twistor w corresponding to a curve T. If $T' \in \partial CV_N$ satisfies $l_{T'}(w) > 0$, then $\lim_{k\to\pm\infty} T'\tau^k = T$.*

If T' satisfies the hypotheses of Proposition 3.2, then we simply say that T' *intersects* T. We call τ as above the Dehn twist associated to the curve T; in light of Proposition 3.2, the ambiguity of replacing τ with τ^{-1} is not important. Notice that if T' is a curve that is disjoint from T, then $T'\tau = T'$. If τ is the Dehn twist associated to a curve T, then we write $\tau = \tau(T)$; dually, we define T_τ to be the unique curve satisfying $\tau(T_\tau) = \tau$.

3.3 Forcing Arational Limits

Here is the first part of our adaptation of Gabai's procedure; we use this result to construct arational trees. Throughout this section, we blur the distinction between a factor and its representatives, arguing with subgroups and their conjugacy classes as needed.

Proposition 3.3. *Let F be a factor, and let T and T' be disjoint curves with neighborhoods $T \in U$, $T' \in U'$. There are disjoint curves T_1, T_1' with neighborhoods $T_1 \in U_1 \subseteq U$, $T_1' \in U_1' \subseteq U'$, such that for any $S \in U_1 \cup U_1'$, S_F is free and simplicial.*

Proof. We will do the proof in the case where both T and T' are splittings with one loop-edge, and the common refinement is a graph with one vertex and two loop edges. The other cases are similar, are easier, and are left as exercises to the reader.

Let V and V' be the vertex groups of T and T', respectively, and let A be the vertex group of the refinement. Let w and w' be the generators of the edge groups of T and T', respectively, and let t and t' be the respective stable letters. Let $A = \langle a_1, \ldots, a_{N-3}, w', w^t, w'^{t'}\rangle$ with $w \in \langle a_1, \ldots, a_{N-3}\rangle$, and where $\{a_1, \ldots, a_{N-3}, t, t', w'\}$ is a basis for F_N. Note that $V = \langle A, t'\rangle$ and $V' = \langle A, t\rangle$.

It is not the case that F can contain V or V'; indeed, both V and V' strictly contain corank 1 factors.

We need to modify T, T' to get new curves in U, U', respectively, so that F does not intersect the vertex groups of the new curves. We accomplish this in several steps, each of which removes certain kinds of intersections; we will appeal to Proposition 3.2 to move curves into U, U'.

First suppose that F contains (after choosing a conjugacy representative) $\langle a_1, \ldots, a_{N-3}, t\rangle$. Note that F cannot also contain both w' and t', or else $F = F_N$. Let ϵ be whichever of these letters is not contained in F. Let f be the automorphism that sends $a_1 \mapsto a_1\epsilon$ and that is the identity on the other basis elements. Replace T with the tree Tf^{-1}, and replace a_1 by $f(a_1)$; note that Tf^{-1} satisfies the hypotheses of the proposition. Furthermore, F does not contain $\langle a_1, \ldots, a_{N-3}, t\rangle$.

Step 1: By Howson's Theorem (see [42]), there are only finitely many conjugacy classes of intersection of F with $\langle a_1, \ldots, a_{N-3}, t\rangle$. Hence, by applying a sufficiently high power of a fully irreducible automorphism on the factor $\langle a_1, \ldots, a_{N-3}, t\rangle$, say ϕ, and extending ϕ to F_N by sending $w' \mapsto w'$ and $t' \mapsto t'$, we can guarantee that in the tree $T\phi^{-1} := T_{1/2}$, conjugates of F intersect the vertex group $\phi(V) = V_{1/2} = \langle \phi(a_1), \ldots, \phi(a_{N-3}), w', t', \phi(w^t)\rangle$ only in elements which contain instances of w', t', and $\phi(w^t)$ (i.e., no element in the intersection can be contained in any subfactor of $\phi\langle a_1, \ldots, a_{N-3}, t\rangle$).

The edge group of $T_{1/2}$ is generated by $\phi(w)$, whose reduced form must be a word containing some instances of t, and hence it is hyperbolic in T. Also, by construction of ϕ, $T_{1/2}$ remains commonly refined with T'. The vertex group of the refinement is $\phi(A) = A_{1/2}$.

By Proposition 3.2, we can apply a high power of the Dehn twist $\tau = \tau(T)$ to $T_{1/2}$ to move $T_{1/2}$ into U; we use $T_{1/2}$ to denote this new curve as well; note that by applying τ, we could not introduce new intersections of F with $V_{1/2}$ that meet V nontrivially, as V is fixed by τ.

Step 2: Now we perturb the tree T'. Consider an automorphism ϕ of the factor $\langle w', t', t\rangle$ which sends $w' \mapsto w'\phi(t)$, $t' \mapsto t'\phi(t)$ and $\phi(t) \mapsto \phi(t)$. Extend ϕ to F_N. Since the intersection of any conjugate of F with $V_{1/2}$ cannot contain $\phi(t)$, it follows that conjugates of F intersect the $A_{1/2}$ only in elements whose reduced form must contain instances of $\phi(w^t)$ (that is, any elements in the intersection of F with $\langle \phi(a_1), \ldots, \phi(a_{N-3}), \phi(w'), \phi(w^t), \phi(w'^{t'})\rangle$ cannot be contained in $\langle \phi(a_1), \ldots, \phi(a_{N-3}), \phi(w'), \phi(w'^{t'})\rangle$).

Denote the tree $T'\phi^{-1}$ by $T'_{1/2}$, so $T'_{1/2}$ is disjoint from $T_{1/2}$. Denote by $A'_{1/2} := \phi(A_{1/2})$ the vertex group of the common refinement.

Step 3: Now we go back to $T_{1/2}$. By the remark at the end of Step 2, if we apply an automorphism g of $V_{1/2} = \langle A'_{1/2}, \phi(t')\rangle$ by $t \mapsto t\phi(t')$ and extending to F_N, in the resulting tree $T_{1/2}g^{-1} = T_1$, no conjugate of F nontrivially intersects the vertex group $V_1 = g(V_{1/2})$.

Again, using Proposition 3.2 with $\tau = \tau(T_{1/2})$, we can move T_1 from the previous paragraph into U; call the new curve T_1 as well. By construction, F does not meet the vertex group of T_1 nontrivially.

Repeat the same process for $T'_{1/2}$ to obtain a tree T'_1 in U', with T'_1 disjoint from T_1 and such that F does not nontrivially intersect the vertex group of T'_1.

To finish we apply Lemma 3.2 along with the fact that $cv(F)$ is open in $\overline{cv}(F)$ to find neighborhoods U_1, U_1' with $T_1 \in U_1 \subseteq U$ and $T_1' \in U_1' \subseteq U'$, as desired.

□

3.4 Forcing Nongeometric Limits

In this section, we bring the second part of our adaptation of Gabai's procedure; we use the main result of this section to construct nongeometric trees as limits of curves. The reader is assumed to be familiar with Rips theory [6]. First, we record the following:

Lemma 3.3. *Suppose that $Y(T, x_T, n)$ contains a nonannular family of width w. For any $\epsilon > 0$, there is $\delta > 0$ such that if T' is δ-close to T, then $Y(T', x_{T'}, n)$ contains a nonannular family of width $w - \epsilon$.*

Proof. Let $\mathcal{C}$ be a nonannular family of width w. Suppose $\mathcal{C}$ intersects $K(T, x_T, n)$ in the intervals $b_0, b_1, b_2, \ldots, b_k$. Use $\phi_{i,j}$ to denote the partial isometry, which corresponds to an element of $\mathscr{B}^{\pm}$, that maps b_i to b_j.

As $\mathcal{C}$ is a family, the interior of each b_i is disjoint from the set of extremal points of $\mathrm{dom}(\phi_b)$, for every partial isometry ϕ_b corresponding to $b \in \mathscr{B}^{\pm}$; further, the orbit of each point of b_i is finite and is contained in $\mathcal{C}$, and the length of each b_i is w. Note that by definition of the Gromov-Hausdorff topology, for any $\eta > 0$, there is $\delta' > 0$, such that for T' δ'-close to T, there is a $(1+\eta)$ bi-Lipschitz, B_n-equivariant map $f = f(T, T', n)$ from an η-dense subtree of $K(T, x_T, n)$ onto an η-dense subtree of $K(T', x_{T'}, n)$; this uses that the $K(\cdot,\cdot,\cdot)$'s are trees.

Now, choose η small enough so that $2k\eta(1+\eta) << \epsilon$, and let T' be δ'-close to T with δ' as in the previous paragraph. Use b_i' for the f-image of b_i, $\phi_{i,j}'$ for the corresponding partial isometries of $K(T', x_{T'}, n)$. Note that $\phi_{i,i+1}'$ is defined on a central segment I_i' of b_i' of length at least $w/(1+\eta) - 2\eta(1+\eta)$ and that $\phi_{i,i+1}'(I_i)$ overlaps with b_{i+1}' in a central segment of width at least $w/(1+\eta) - 4\eta(1+\eta)$. Hence, there is a central segment $J_0 \subseteq b_0'$ of length at least $w/(1+\eta) - 2k\eta(1+\eta)$ with $\phi_{0,j}'(J_0)$ contained in the central segment of length $w/(1+\eta) - 2\eta(1+\eta)$ of b_j'. Hence, no $\phi_{0,j}'$-image of J_0 can meet an extremal point of $\mathrm{dom}(\phi_{i,i+1}')$.

Let $\mathcal{C}'$ be the union of leaves in $Y(T', x_{T'}, n)$ containing the points of J_0. Note that by choosing T' δ'-close to T, we have ensured that $\phi_{0,1}'$ is the only partial isometry from $\mathscr{B}^{\pm}$ that is defined on J_0, since this is true in $Y(T, x_T, n)$; similarly, $(\phi_{k-1,k}')^{-1}$ is the only element of $\mathscr{B}^{\pm}$ that is defined on $\phi_{0,k}'(J_0)$. Hence, $\mathcal{C}'$ is contained in a family, which has width at least $w - \epsilon$, as desired. □

Proposition 3.4. *Let T and T' be disjoint curves with neighborhoods $T \in U$, $T' \in U'$, and let $n \in \mathbb{N}$ be given. There are disjoint curves T_1, T_1' with neighborhoods $T_1 \in U_1 \subseteq U$, $T_1' \in U_1' \subseteq U'$, such that for any $S \in U_1 \cup U_1'$, $Y(S, x_S, n)$ contains a nonannular family of width bounded away from zero.*

Proof. We begin with an observation: if $A \in \partial\mathrm{cv}_N$ and if $y \in A$ is a point that is fixed by a subgroup $H \leq F_N$, then H fixes a point in A_n if and only if $H^g = \langle h_1, \ldots, h_r \rangle$ for $\{h_1, \ldots, h_r\} \subseteq B_n$, where $g \in F_N$ is chosen so that H^g fixes $gy \in K(A, x_a, 1)$ (H^g is cyclically reduced with respect to $\mathscr{B}$). It follows that if A is the curve with vertex group $V = \langle a_1, \ldots, a_{N-2}, w, w^t \rangle$, edge stabilizer $\langle w \rangle$, and stable letter t, then A_n contains an edge with nontrivial stabilizer if and only if $w \in H$ and $w^t \in H_t$, with H, H_t fixing points in A and having generating sets contained in B_n.

In light of the discussion in the above paragraph, we proceed along the lines of the proof of Proposition 3.3. For simplicity, we assume that $\mathscr{B} = \{a_1, \ldots, a_{N-3}, w', t, t'\}$ is a basis for F_N so that the vertex groups V, V' of T, T' are $V = \langle a_1, \ldots, a_{N-3}, w', t', w^t \rangle$, $V' = \langle a_1, \ldots, a_{N-3}, t, w', {w'}^{t'} \rangle$; any other basis gives a quasi-isometric word metric, and the below argument is more cumbersome with arbitrary $\mathscr{B}$.

Replace t with tu, where u is a long random word in $\langle w', t' \rangle$. Next, as in Step 1 of the proof of Proposition 3.3, find an irreducible automorphism ϕ of $\langle a_1, \ldots, a_{N-3}, t \rangle$ with large dilitation and extend ϕ to F_N in the obvious way. Use ψ to denote the composition of these two automorphisms. Apply a high power of $\tau(T)$ to $T\psi$ to get $T_{1/2} \in U$. Note that the stable letter for T_1 has very long length with respect to $\mathscr{B}$.

Now, repeat the procedure from the above paragraph to T' to get $T'_{1/2}$ in U' with stable letter having very long length with respect to $\mathscr{B}$. Now repeat both these operations on $T_{1/2}$ and $T'_{1/2}$ to get T_1 and T_1'.

By the discussion in the first paragraph of this proof, there is $w > 0$ so that both $Y(T_1, x_{T_1}, n)$ and $Y(T_1', x_{T_1'}, n)$ contain a nonannular family of width at least w. Choose $\epsilon << w$, and let $\delta > 0$ be as given by Lemma 3.3. The intersections of the δ-balls around T_1, T_1' with U, U' give neighborhoods U_1, U_1' of T, T' satisfying the conclusions of the statement. $\square$

3.5 The Proof of the Main Result

We will use Lemma 3.3 along with the following characterization of nongeometric trees with dense orbits. Note that arational trees have dense orbits [36].

Lemma 3.4. *Let $T \in \partial cv_N$ have dense orbits. The tree T is nongeometric if and only if $Y(T, x_T, n)$ contains a nonannular family for every n.*

Proof. By Imanishi's theorem, if T has dense orbits and is geometric, then for $n >> 0$, $Y(T, x_T, n)$ is a union of minimal components. Further, the space $Y(T, x_T, n)$ can contain an annular family only if T contains a nondegenerate arc with nontrivial stabilizer, which is impossible if T has dense orbits; see, for instance, [30]. $\square$

Having $T_n \to T$ not exact can be thought of as T being nongeometric on the scale B_n. Our interest in Lemmas 3.3 and 3.4 can be paraphrased as follows for trees T with dense orbits: if T is nongeometric on scale B_n, then for T' close enough to T, T' also is nongeometric on scale B_n; and if T is nongeometric on the scale B_n for every n, then T is nongeometric. Now, we are in position to prove our main result.

Theorem 3.5. *Let T, T' be disjoint curves with neighborhoods U, U'. There is a 1-simplex of nonuniquely ergodic, arational, nongeometric trees with one extreme point in each of U, U'.*

Proof. Enumerate all factors of F_N as $F^1, F^2, \ldots, F^k, \ldots$. Set $T_0 = T$, $T_0' = T'$ and $U_0 = U$, $U_0' = U'$. We proceed inductively, defining for $k > 0$ T_k, T_k' with neighborhoods $U_k \subseteq U_l$, $U_k' \subseteq U_l'$ for $l \leq k$, with $\{F^j\}_{j \leq l}$ acting freely and simplicially on any $S \in U_k \cup U_k'$ and with $Y(S, x_S, l)$ containg a nonannular family of width greater than $w(k) > 0$ for any $S \in U_k \cup U_k'$. Assume that T_{k-1}, T_{k-1}' are defined. To define T_k, T_k', U_k, U_k', first apply Proposition 3.3 to T_{k-1}, T_{k-1}', U_{k-1}, U_{k-1}', and then apply Proposition 3.4 to the result. By shrinking the neighborhoods from the conclusion of Proposition 3.3 slightly, we can assume that they are contained in compact neighborhoods satisfying the same conclusions.

Note that each U_k, U_k' is contained in a ball by construction, and the radii of these balls must go to zero (for example, by density of nonarational trees in $\partial \mathrm{cv}_N$). Hence, T_k converge as to some $\lambda \in \partial \mathrm{cv}_N$, and T_k' converge to some λ'. By construction, any factor of F_N acts freely and simplicially on λ and λ'. Further, by Lemma 3.4, λ and λ' are nongeometric.

Use w_k to denote the edge stabilizer of T_k, and let η_k be the counting current corresponding to w_k, so $\langle T_k, \eta_k \rangle = 0$. Since T_k' is disjoint from T_k, we have that $\langle T_k', \eta_k \rangle = 0$ as well. Let η be a representative of any accumulation point in projective current of the images of η_k. By continuity of $\langle \cdot, \cdot \rangle$, we have that $\langle \lambda, \eta \rangle = 0 = \langle \lambda', \eta \rangle$. Further, by the Kapovich-Lustig characterization of zero intersection, we have that $\emptyset \neq Supp(\eta) \subseteq L(\lambda) \cap L(\lambda')$.

By Theorem 4.4 of [10], we have that $L(\lambda) = L(\lambda')$, and by Theorem II of [14], any convex combination $\alpha l_\lambda + (1 - \alpha) l_{\lambda'}$ is the length function of a very small tree T_α. On the other hand, from the definition of $L(\cdot)$, we certainly have that $L(\lambda) \subseteq L(T_\alpha)$; applying Theorem 4.4 of [10] again gives that T_α is arational with $L(T_\alpha) = L(\lambda)$. Hence, the segment $\{T_\alpha | \alpha \in [0,1]\}$ satisfies the conclusion. $\square$

CHAPTER 4

SOME OPEN QUESTIONS

Here we list some interesting (to the author) open questions. Each of this questions aims to show a basic, but yet unproven, fact about some of the complexes defined above.

4.1 Is the Automorphism Group of the Free Factor Complex Isomorphic to $Out(F_N)$?

This question is known for the the spine of CV_N [12] and for FS_N [1]. The proof for FS_N uses strongly that FS_N sits in the boundary of CV_N as the "missing simplices." Of course, such an argument will not work for the free factor complex.

4.2 Is the Free Factor Complex FF_N Homotopy Equivalent to the Wedge of $(N-2)$-Spheres?

It is known that the Tits Building for $SL_n(\mathbb{Z})$ has the homotopy type of a wedge of $n-2$ spheres. Similarly, it is shown in [25] that the complex whose vertices are actual free factors of F_N, and not conjugacy classes and whose edges are given by inclusion is also homotopy equivalent to a wedge of $(N-2)$-spheres.

Is the same result true for FF_N?

Bestvina gives a simple argument for the $SL_n(\mathbb{Z})$ using PL Morse theory in [2], and the argument in [25] is essentially Morse theoretic in that it studies descending links of vertices in order to "build up" the entire space from these links, and then uses a spectral sequence of Quillen to deduce that this factor complex has the right homotopy type. Does some Morse theory argument work for FF_N as well?

4.3 Which Elements of $Out(F_N)$ Act Hyperbolically on FZ_N?

It is known that pseudo-Anasov mapping classes are the hyperbolic automorphisms of the curve complex, and furthermore, that fully irreducible elements of $Out(F_N)$ are exactly those which act hyperbolically on FF_N.

Handel-Mosher claim that the class of automorphisms which act hyperbolically on FS_N are exactly those which have a filling attracting lamination.

Which class of automorphisms acts hyperbolically on the cyclic splitting complex?

4.4 Does the Action of $Out(F_N)$ on FZ_N Satisfy Weak Proper Discontinuity?

All the examples known where WPD fails for FS_n are coned to a finite diameter set in FZ_n.

Example 4.1. *Let $R = R_{k+1}$ be a rose and S a surface of genus g with one boundary component. Label the loops of R_{k+1} by $a_1, a_2, \ldots, a_{k+1}$. Let $X = R \cup_{\partial S = a_{k+1}} S$, so that $\pi_1(X) \cong F_{2g+k}$.*

Let $\phi \in Out(F_{2g+k})$ be an automorphism whose restriction to R is fully irreducible. Since a_{k+1} represents the boundary word of S, perhaps choosing ϕ more carefully, we see that ϕ has some attracting lamination which fills F_{2g+k}, and hence acts hyperbolically on FS_n.

However, the stabilizer of the axis of ϕ in FS_n is not virtually cyclic; indeed, take any pseudo-Anasov on S. This fixes the boundary word of S, so it commutes with ϕ, but is not even virtually cyclic.

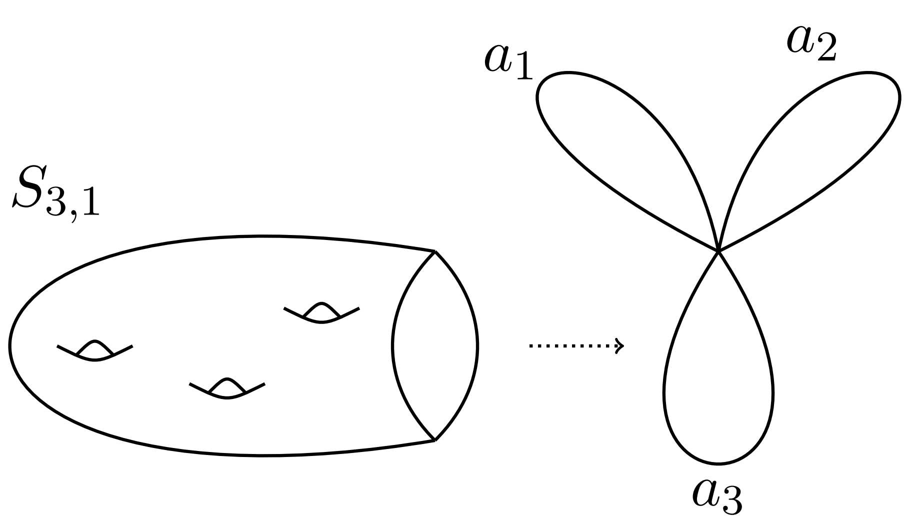

Figure 4.1. A picture of what is going on in the above text. The dotted arrow represents the attachment $\partial S_{3,1} \cong a_3$.

REFERENCES

[1] Javier Aramayona and Juan Souto. Automorphisms of the graph of free splittings. *arXiv.org:0909.3660v1*, 2009.

[2] Mladen Bestvina. PL morse theory. *Mathematical Communications*, 13:149–162, 2008.

[3] Mladen Bestvina. A Bers-like proof of the existence of train tracks for free group automorphisms. *arXiv:1001.0325v2*, 2011.

[4] Mladen Bestvina. PCMI lectures on the geometry of outer space, 2012.

[5] Mladen Bestvina and Mark Feighn. Outer limits (preprint). *http://andromeda.rutgers.edu/ feighn/papers/outer.pdf*, 1994.

[6] Mladen Bestvina and Mark Feighn. Stable actions of groups on real trees. *Invent. Math.*, 121(2):287–321, 1995.

[7] Mladen Bestvina and Mark Feighn. A hyperbolic $Out(F_N)$ complex. *arXiv.org:0808.3730v4*, 2009.

[8] Mladen Bestvina and Mark Feighn. Hyperbolicity of the complex of free factors. *arXiv:1107.3308*, 2011.

[9] Mladen Bestvina and Koji Fujiwara. Bounded cohomology of subgroups of mapping class groups. *Geometry and Topology*, 6:69–89, 2002.

[10] Mladen Bestvina and Patrick Reynolds. The boundary of the complex of free factors (preprint). *arXiv:1211.3608*, 2012.

[11] B Bowditch. Intersection numbers and the hyperbolicity of the curve complex. *J. Reine Angew. Math.*, 598:105–129, 2006.

[12] Martin R. Bridson and Karen Vogtmann. The symmetries of outer space. *Duke Math. J*, 106:391–409, 2001.

[13] Marshall Cohen and Martin Lustig. Very small group actions on R-trees and Dehn twist automorphisms. *Topology*, 34:575–617, 1995.

[14] Thierry Coulbois, Arnaud Hilion, and Martin Lustig. Non-unique ergodicity, observers' topology and the dual algebraic lamination for $\mathbb{R}$-trees. *Illinois J. Math.*, 51(3):897–911, 2007.

[15] Thierry Coulbois, Arnaud Hilion, and Martin Lustig. $\mathbb{R}$-trees and laminations for free groups. II. the dual lamination of an $\mathbb{R}$-tree. *J. Lond. Math. Soc. (2)*, 78(3):737–754, 2008.

[16] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Inventiones Mathematicae*, 84:91–119, 1986.

[17] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups.* Princeton University Press, 2011.

[18] Albert Fathi, Francois Laudenbach, and Valentin Poenaru. *Thurston's Work on Surfaces.* Princeton University Press, 2012.

[19] David Gabai. Almost filling laminations and connectivity of ending lamination space. *Geom. Top.*, 13:1017–1041, 2009.

[20] Damien Gaboriau, Gilbert Levitt, and Fréderic Paulin. Pseudogroups of isometries of $\mathbb{R}$ and Rips' theorem on free actions on $\mathbb{R}$-trees. *Israel J. Math.*, 87:403–428, 1994.

[21] Vincent Guirardel. Approximations of stable actions on $\mathbb{R}$-trees. *Commentarii Mathematici Helvetici*, 73(1), 1998.

[22] Vincent Guirardel. Dynamics of $\mathrm{Out}(F_n)$ on the boundary of outer space. *Ann. Sci. École Norm. Sup.*, 33(4):433–465, 2000.

[23] Ursula Hamenstaedt. The boundary of the free splitting graph and the free factor graph. *arXiv:1211.1630*, 2013.

[24] Michael Handel and Lee Mosher. The free splitting complex of a free group I: Hyperbolicity. *arXiv:1111.1994*, 2011.

[25] Allen Hatcher and Karen Vogtmann. The complex of free factors of a free group. *Quart. J. Math. Oxford Ser*, 2:459–468, 1998.

[26] Arnaud Hilion and Camille Horbez. The hyperbolicity of the sphere complex via surgery paths. *arXiv:1210.6183*, 2012.

[27] Ilya Kapovich and Martin Lustig. Geometric intersection number and analogues of the curve complex for free groups. *Geometry and Topology*, 13:1805–1833, 2009.

[28] Ilya Kapovich and Martin Lustig. Intersection form, laminations and currents on free groups. *Geom. Funct. Anal.*, 19(5):1426–1467, 2010.

[29] Ilya Kapovich and Kasra Rafi. Hyperbolicity of the free factor complex. *arXiv:1206.3626*, 2012.

[30] Gilbert Levitt and Martin Lustig. Irreducible automorphisms of F_N have north-south dynamics on compactified outer space. *J. Inst. Math. Jussieu*, 2:59–72, 2003.

[31] Reiner Martin. Measured currents and automorphisms of free groups. preprint, 1995.

[32] Reiner Martin. Non-uniquely ergodic foliations of thin type. *Ergodic Theory Dynam. Systems*, 17(3):667–674, 1997.

[33] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves I: Hyperbolicity. *Inventiones Mathematicae*, 138:103–149, 1999.

[34] John Morgan and Peter Shalen. Valuations, trees, and degenerations of hyperbolic structures, I. *Annals of Mathematics*, 1984.

[35] Patrick Reynolds. Dynamics of irreducible endomorphisms of F_n (preprint). *arXiv:1008.3659*, 2010.

[36] Patrick Reynolds. Reducing systems for very small trees (preprint). *arXiv:1211.3378*, 2012.

[37] Abe Shenitzer. Decomposition of a group with one defining relation into a free product. In *Proceedings of the American Mathematical Society*, 1955.

[38] Richard Skora. Deformations of length functions in groups. *pre-print*, 1990.

[39] Richard Skora. Splittings of surfaces. *J. Amer. Math. Soc.*, 9:605–616, 1996.

[40] J Stallings. Topology of finite graphs. *Inventiones Mathematicae*, 71:551–565, 1983.

[41] John Stallings. Foldings of G-trees. In Roger C. Alperin, editor, *Arboreal Group Theory*, pages 355–368. Mathematical Sciences Research Institute, 1988.

[42] John R. Stallings. Topology of finite graphs. *Invent. Math.*, 71(3):551–565, 1983.

[43] Gadde A. Swarup. Decompositions of free groups. *J. Pure. Appl. Algebra*, 40:99–102, 1986.